Chapter 2 :
Conservation Laws of Fluid Motion and Boundary Conditions

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Governing Equations of Fluid Flow and Heat Transfer

The governing equations of fluid flow represent mathematical statements of the **conservation laws of physics**.

- The mass of fluid is conserved
- The rate of change of momentum equals the sum of the forces on a fluid particle (Newton's second law)
- The rate of change of energy is equal to the sum of the rate of heat addition to and the rate of work done on a fluid particle (first law of thermodynamics).

The six faces are labelled N, S, E, W, T, B

The center of the element is located at position (x, y, z)

$\rho = \rho\,(x, y, z, t) \qquad p = p\,(x, y, z, t)$

$T = T\,(x, y, z, t) \qquad \boldsymbol{u} = \boldsymbol{u}\,(x, y, z, t)$

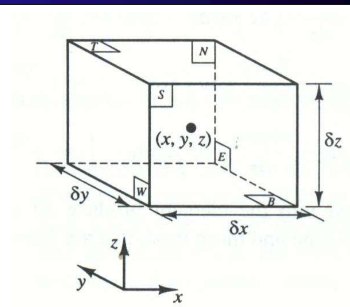


Fig. 2-1 Fluid element for conservation laws.

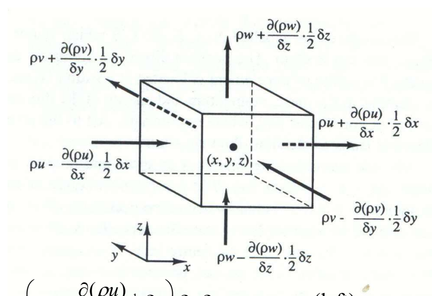
Fluid properties at faces are approximated by means of the two terms of the Taylor series

The pressure at the W and E faces, can be expressed as

$$p - \frac{\partial p}{\partial x}\tfrac{1}{2}\delta x \qquad \text{and} \qquad p + \frac{\partial p}{\partial x}\tfrac{1}{2}\delta x$$

Mass Conservation in Three Dimensions

$$\begin{pmatrix}\text{Rate of increase}\\ \text{of mass in}\\ \text{fluid element}\end{pmatrix} = \begin{pmatrix}\text{Net rate of flow}\\ \text{of mass into}\\ \text{fluid element}\end{pmatrix}$$



$$\frac{\partial}{\partial t}(\rho\delta x\delta y\delta z) = \frac{\partial\rho}{\partial t}(\delta x\delta y\delta z) = \left(\rho u - \frac{\partial(\rho u)}{\partial x}\tfrac{1}{2}\delta x\right)\delta y\delta z - \left(\rho u + \frac{\partial(\rho u)}{\partial x}\tfrac{1}{2}\delta x\right)\delta y\delta z \qquad \text{(left)}$$

$$+\left(\rho v - \frac{\partial(\rho v)}{\partial y}\tfrac{1}{2}\delta y\right)\delta x\delta z - \left(\rho v + \frac{\partial(\rho v)}{\partial y}\tfrac{1}{2}\delta y\right)\delta x\delta z \qquad \text{(front)}$$

$$+\left(\rho w - \frac{\partial(\rho w)}{\partial z}\tfrac{1}{2}\delta z\right)\delta x\delta y - \left(\rho w + \frac{\partial(\rho w)}{\partial z}\tfrac{1}{2}\delta z\right)\delta x\delta y \qquad \text{(bottom)}$$

$$\frac{\partial\rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z} = 0$$

Or in more compact vector notation → $$\frac{\partial\rho}{\partial t} + \overbrace{div(\rho\mathbf{u})}^{\text{Net flow of mass out of the control volume}} = 0 \qquad \text{(2-4)}$$

For an incompressible fluid ρ = const → $div(\rho\mathbf{u}) = 0$ or $\dfrac{\partial(\rho u)}{\partial x} + \dfrac{\partial(\rho v)}{\partial y} + \dfrac{\partial(\rho w)}{\partial z} = 0$

Rates of change following a fluid particle and for a fluid element

The total or substantial derivative of ϕ with respect to time following a fluid particle is

$$\frac{D\phi}{Dt} = \frac{\partial\phi}{\partial t} + \frac{\partial\phi}{\partial x}\frac{dx}{dt} + \frac{\partial\phi}{\partial y}\frac{dy}{dt} + \frac{\partial\phi}{\partial z}\frac{dz}{dt}$$

A fluid particle follows the flow, so

$$dx/dt = u$$
$$dy/dt = v$$
$$dz/dt = w$$

Hence the substantive derivative of ϕ is given by

$$\frac{D\phi}{Dt} = \frac{\partial\phi}{\partial t} + u\frac{\partial\phi}{\partial x} + v\frac{\partial\phi}{\partial y} + w\frac{\partial\phi}{\partial z} = \frac{\partial\phi}{\partial t} + \mathbf{u}\cdot grad\,\phi$$

$D\phi/Dt$ defines the rate of change of property ϕ per unit mass.

The rate of change of property ϕ per unit volume for a fluid particle is $\rho D\phi/Dt$, hence

$$\rho\frac{D\phi}{Dt} = \rho\left(\frac{\partial\phi}{\partial t} + \mathbf{u}\cdot grad\,\phi\right) \qquad (2\text{-}8)$$

ρ = mass per unit volume.

Lhs of the mass conservation equation (2-4) is

$$\frac{\partial\rho}{\partial t} + div(\rho\mathbf{u})$$

The generalization of these terms for an arbitrary conserved property is

$$\frac{\partial(\rho\phi)}{\partial t} + div(\rho\phi\mathbf{u}) \qquad (2\text{-}9)$$

$$\begin{pmatrix}\text{Rate of increase}\\ \text{of } \phi \text{ per unit volume}\end{pmatrix} + \begin{pmatrix}\text{Net rate of flow of } \phi\\ \text{out of fluid element}\\ \text{per unit volume}\end{pmatrix}$$

Rewriting eq. (2-9)

$$\frac{\partial(\rho\phi)}{\partial t}+div(\rho\phi\mathbf{u})=\rho\left[\frac{\partial\phi}{\partial t}+\mathbf{u}\cdot grad\,\phi\right]+\phi\overbrace{\left[\frac{\partial\rho}{\partial t}+div(\rho\mathbf{u})\right]}^{=0\text{ (due to conservation of mass)}}=\rho\frac{D\phi}{Dt} \qquad (2\text{-}10)$$

$$\begin{pmatrix}\text{Rate of increase}\\ \text{of }\phi\text{ of}\\ \text{fluid element}\end{pmatrix}+\begin{pmatrix}\text{Net rate of flow of }\phi\\ \text{out of}\\ \text{fluid element}\end{pmatrix}=\begin{pmatrix}\text{Rate of increase}\\ \text{of }\phi\text{ for a}\\ \text{fluid particle}\end{pmatrix}$$

Relevant entries of ϕ for momentum and energy equations as defined in Eqn (2.10) are:

x-momentum	u	$\rho\frac{Du}{Dt}$	$\frac{\partial(\rho u)}{\partial t}+div(\rho u\mathbf{u})$
y-momentum	v	$\rho\frac{Dv}{Dt}$	$\frac{\partial(\rho v)}{\partial t}+div(\rho v\mathbf{u})$
z-momentum	w	$\rho\frac{Dw}{Dt}$	$\frac{\partial(\rho w)}{\partial t}+div(\rho w\mathbf{u})$
energy	E	$\rho\frac{DE}{Dt}$	$\frac{\partial(\rho E)}{\partial t}+div(\rho E\mathbf{u})$

The rates of increase of x-, y-, and z-momentum per unit volume are

$$\rho\frac{Du}{Dt} \qquad \rho\frac{Dv}{Dt} \qquad \rho\frac{Dw}{Dt}$$

We distinguish two types of forces on fluid particles:

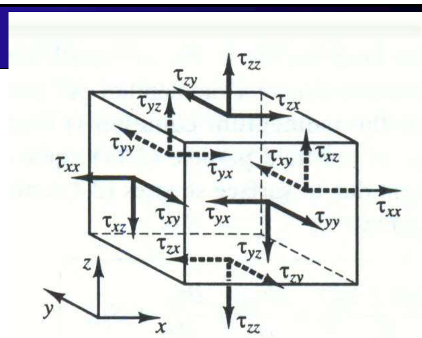
- **surface forces** - pressure forces
  - viscous forces
- **body forces** - gravity forces
  - centrifugal forces
  - Coriolis forces
  - electromagnetic force

source terms

The pressure, a normal stress, is denoted by p.

Viscous stresses are denoted by τ.

Fig. 2-3 Stress components on three faces of fluid element.



The suffices i and j in τ_{ij} indicate that the stress component acts in the j-direction on a surface normal to the i-direction.

First we consider the x-components of the forces due to pressure p and stress components τ_{xx}, τ_{yx} and τ_{zx} shown in Fig. 2-4.

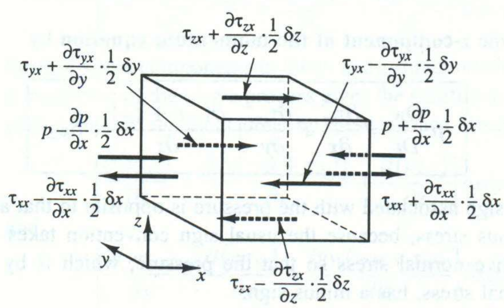
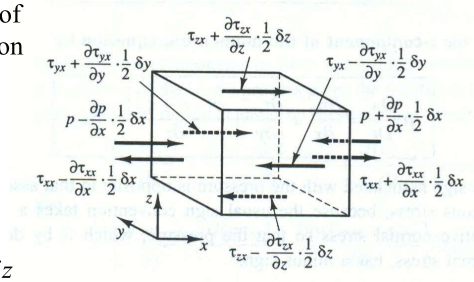


Fig. 2-4 Stress components in the x-direction.

The net force in the x-direction is the sum of the force components acting in that direction on the fluid element.



On the pair of faces (E, W) we have

$$\left[\left(\cancel{p} - \frac{\partial p}{\partial x}\tfrac{1}{2}\delta x\right) - \left(\cancel{\tau_{xx}} - \frac{\partial \tau_{xx}}{\partial x}\tfrac{1}{2}\delta x\right)\right]\delta y \delta z$$
$$+\left[-\left(\cancel{p} + \frac{\partial p}{\partial x}\tfrac{1}{2}\delta x\right) + \left(\cancel{\tau_{xx}} + \frac{\partial \tau_{xx}}{\partial x}\tfrac{1}{2}\delta x\right)\right]\delta y \delta z$$
$$= \left(-\frac{\partial p}{\partial x} + \frac{\partial \tau_{xx}}{\partial x}\right)\delta x \delta y \delta z \qquad (2\text{-}12a)$$

The net force in the x-direction on the pair of faces (N, S) is

$$-\left(\tau_{yx} - \frac{\partial \tau_{yx}}{\partial y}\tfrac{1}{2}\delta y\right)\delta x \delta z + \left(\tau_{yx} + \frac{\partial \tau_{yx}}{\partial y}\tfrac{1}{2}\delta y\right)\delta x \delta z = \frac{\partial \tau_{yx}}{\partial y}\delta x \delta y \delta z \qquad (2\text{-}12b)$$

The net force in the x-direction on the pair of faces (T, B) is

$$-\left(\tau_{zx} - \frac{\partial \tau_{zx}}{\partial z}\tfrac{1}{2}\delta z\right)\delta x \delta y + \left(\tau_{zx} + \frac{\partial \tau_{zx}}{\partial z}\tfrac{1}{2}\delta z\right)\delta x \delta y = \frac{\partial \tau_{zx}}{\partial z}\delta x \delta y \delta z \qquad (2\text{-}12c)$$

The total force per unit volume on the fluid due to these surface stresses is equal to the sum of (2-12a), (2-12b), (2-12c) divided by the volume $\delta x \delta y \delta z$:

$$\frac{\partial(-p+\tau_{xx})}{\partial x}+\frac{\partial \tau_{yx}}{\partial y}+\frac{\partial \tau_{zx}}{\partial z} \qquad (2.13)$$

To find x-component of the momentum equation:

$$\underbrace{\begin{pmatrix}\text{Rate of change of}\\ x\text{-momentum of}\\ \text{fluid particle}\end{pmatrix}}_{\text{Eqn.(2.11)}} = \underbrace{\begin{pmatrix}\text{Total force in } x\text{-direction}\\ \text{on the element due to}\\ \text{surface stresses}\end{pmatrix}}_{\text{Eqn.(2.13)}} + \underbrace{\begin{pmatrix}\text{Total force in } x\text{-direction}\\ \text{on the element due to}\\ \text{body forces}\end{pmatrix}}_{S_{Mx}}$$

$$\rho\frac{Du}{Dt}=\frac{\partial(-p+\tau_{xx})}{\partial x}+\frac{\partial \tau_{yx}}{\partial y}+\frac{\partial \tau_{zx}}{\partial z}+S_{Mx} \qquad (2.14a)$$

S_{Mx} = Body force on the element per unit volume in x-direction

$S_{Mz} = -\rho g$ (body force due to gravity per unit volume in z-direction)

Similarly, y-component of the momentum equation is

$$\rho\frac{Dv}{Dt}=\frac{\partial \tau_{xy}}{\partial x}+\frac{\partial(-p+\tau_{yy})}{\partial y}+\frac{\partial \tau_{zy}}{\partial z}+S_{My} \qquad (2.14b)$$

and, z-component of the momentum equation is

$$\rho\frac{Dw}{Dt}=\frac{\partial \tau_{xz}}{\partial x}+\frac{\partial \tau_{yz}}{\partial y}+\frac{\partial(-p+\tau_{zz})}{\partial z}+S_{Mz} \qquad (2.14c)$$

Energy Equation in Three Dimensions

The energy equation is derived from the **first law of thermodynamics** which states that

$$\underbrace{\begin{pmatrix}\text{Rate of increase} \\ \text{of energy of fluid particle}\end{pmatrix}}_{\rho \frac{DE}{Dt}} = \begin{pmatrix}\text{Net rate of} \\ \text{heat added to} \\ \text{fluid particle}\end{pmatrix} + \begin{pmatrix}\text{Net rate of work} \\ \text{done on} \\ \text{fluid particle}\end{pmatrix}$$

Work Done by Surface Forces = $(F_{\text{surface forces}})(V)$

V = velocity component in the direction of the force.

The surface forces given by (2.12a-c) all act in the x-direction.

The net rate of work done by these forces acting in x-direction is

$$\left[\frac{\partial\left[u(-p+\tau_{xx})\right]}{\partial x} + \frac{\partial(u\tau_{yx})}{\partial y} + \frac{\partial(u\tau_{zx})}{\partial z}\right]\delta x\delta y\delta z \qquad (2.16a)$$

Similarly, work done by surface stresses in y and z-direction are

$$\left[\frac{\partial(v\tau_{xy})}{\partial x} + \frac{\partial\left[v(-p+\tau_{yy})\right]}{\partial y} + \frac{\partial(v\tau_{zy})}{\partial z}\right]\delta x\delta y\delta z \qquad (2.16b)$$

$$\left[\frac{\partial(w\tau_{xz})}{\partial x} + \frac{\partial(w\tau_{yz})}{\partial y} + \frac{\partial\left[w(-p+\tau_{zz})\right]}{\partial z}\right]\delta x\delta y\delta z \qquad (2.16c)$$

Summing (2.16a-c) yields the **total rate of work done on the fluid particle by surface stresses:**

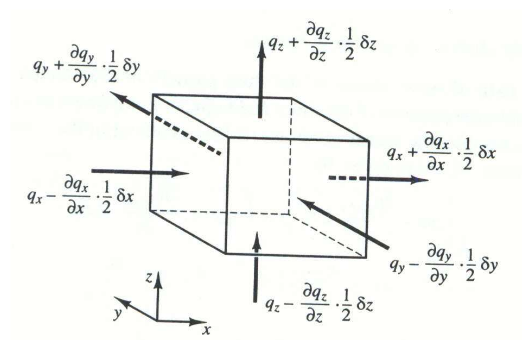
$$\left[-div(p\mathbf{u})\right]+\frac{\partial(u\tau_{xx})}{\partial x}+\frac{\partial(u\tau_{yx})}{\partial y}+\frac{\partial(u\tau_{zx})}{\partial z}+\frac{\partial(v\tau_{xy})}{\partial x}+\frac{\partial(v\tau_{yy})}{\partial y}+\frac{\partial(v\tau_{zy})}{\partial z}$$
$$+\frac{\partial(w\tau_{xz})}{\partial x}+\frac{\partial(w\tau_{yz})}{\partial y}+\frac{\partial(w\tau_{zz})}{\partial z}$$

where

$$-div(p\mathbf{u})=-\frac{\partial(up)}{\partial x}-\frac{\partial(vp)}{\partial y}-\frac{\partial(wp)}{\partial z}$$

Energy Flux due to Heat Conduction

The heat flux vector has three components q_x, q_y, q_z



The net rate of heat transfer to the CV due to heat flow in *x*-direction is

$$\left[\left(\cancel{q_x}-\frac{\partial q_x}{\partial x}\tfrac{1}{2}\delta x\right)-\left(\cancel{q_x}+\frac{\partial q_x}{\partial x}\tfrac{1}{2}\delta x\right)\right]\delta y\delta z=-\frac{\partial q_x}{\partial x}\delta x\delta y\delta z \qquad (2.18b\text{-}c)$$

Similarly, the net rates of heat transfer to the fluid due to heat flows in the y- and z-direction are

$$-\frac{\partial q_y}{\partial y}\delta x\delta y\delta z \qquad \text{and} \qquad -\frac{\partial q_z}{\partial z}\delta x\delta y\delta z \qquad (2.18\text{b-c})$$

The net rate of heat added to CV per unit volume is the sum of (2.18a-c) divided by $\delta x\delta y\delta z$

$$-\frac{\partial q_x}{\partial x}-\frac{\partial q_y}{\partial y}-\frac{\partial q_z}{\partial z}=-div\,\mathbf{q} \qquad (2.19)$$

$$q_x=-k\frac{\partial T}{\partial x} \qquad q_y=-k\frac{\partial T}{\partial y} \qquad q_z=-k\frac{\partial T}{\partial z}$$

This can be written in vector form as

$$\mathbf{q}=-k\,grad\,T \qquad (2.20)$$

Combining (2.19) and (2.20) yields the **rate of heat addition to the CV due to heat conduction**

$$-div\,\mathbf{q}=div(k\,grad\,T)$$

Energy Equation

$$\rho\frac{DE}{Dt}=\overbrace{\left[\begin{array}{l}-div(p\mathbf{u})+\dfrac{\partial(u\tau_{xx})}{\partial x}+\dfrac{\partial(u\tau_{yx})}{\partial y}+\dfrac{\partial(u\tau_{zx})}{\partial z}+\dfrac{\partial(v\tau_{xy})}{\partial x}\\+\dfrac{\partial(v\tau_{yy})}{\partial y}+\dfrac{\partial(v\tau_{zy})}{\partial z}+\dfrac{\partial(w\tau_{xz})}{\partial x}+\dfrac{\partial(w\tau_{yz})}{\partial y}+\dfrac{\partial(w\tau_{zz})}{\partial z}\end{array}\right]}^{\text{sum of the net rate of work done on the CV by surface stresses (2.17)}} \qquad (2.22)$$

$$+\underbrace{div(k\,grad\,T)}_{\text{net rate of heat addition to the fluid (2.21)}}+\underbrace{S_E}_{\text{rate of increase of energy due to sources}}$$

$$E=i+\underbrace{\tfrac{1}{2}(u^2+v^2+w^2)}_{\text{kinetic energy}}$$

$i =$ internal (thermal) energy

S_E = source of energy per unit volume per unit time (i.e. effects of gravitational potential energy changes)

Multiplying

the x-momentum equation (2.14a) by u

the y-momentum equation (2.14a) by v

the z-momentum equation (2.14a) by w

and adding the results together

$$\rho \frac{D\left[\frac{1}{2}(u^2+v^2+w^2)\right]}{Dt} = -\mathbf{u}\cdot grad\ p + u\left(\frac{\partial \tau_{xx}}{\partial x}+\frac{\partial \tau_{yx}}{\partial y}+\frac{\partial \tau_{zx}}{\partial z}\right)$$
$$+v\left(\frac{\partial \tau_{xy}}{\partial x}+\frac{\partial \tau_{yy}}{\partial y}+\frac{\partial \tau_{zy}}{\partial z}\right)$$
$$+w\left(\frac{\partial \tau_{xz}}{\partial x}+\frac{\partial \tau_{yz}}{\partial y}+\frac{\partial \tau_{zz}}{\partial z}\right)+\mathbf{u}\cdot\mathbf{S}_M \qquad (2.23)$$

Subtracting (2.23) from (2.22)

$$\rho\frac{Di}{Dt} = -p\, div\,\mathbf{u} + div(k\, grad\, T) + \tau_{xx}\frac{\partial u}{\partial x}+\tau_{yx}\frac{\partial u}{\partial y} \qquad (2.24)$$
$$+\tau_{zx}\frac{\partial u}{\partial z}+\tau_{xy}\frac{\partial v}{\partial x}+\tau_{yy}\frac{\partial v}{\partial y}+\tau_{zy}\frac{\partial v}{\partial z}+\tau_{xz}\frac{\partial w}{\partial x}$$
$$+\tau_{yz}\frac{\partial w}{\partial y}+\tau_{zz}\frac{\partial w}{\partial z}+S_i$$

where $S_i = S_E - \mathbf{u}.\mathbf{S}_M$

For an incompressible fluid → $i = cT$ and $div\ \mathbf{u} = 0$ (c = specific heat)

$$\rho c\frac{DT}{Dt} = div(k\, grad\, T) + \tau_{xx}\frac{\partial u}{\partial x}+\tau_{yx}\frac{\partial u}{\partial y}+\tau_{zx}\frac{\partial u}{\partial z} \qquad (2.25)$$
$$+\tau_{xy}\frac{\partial v}{\partial x}+\tau_{yy}\frac{\partial v}{\partial y}+\tau_{zy}\frac{\partial v}{\partial z}+\tau_{xz}\frac{\partial w}{\partial x}$$
$$+\tau_{yz}\frac{\partial w}{\partial y}+\tau_{zz}\frac{\partial w}{\partial z}+S_i$$

$$h = i + p/\rho \quad \text{and} \quad h_o = h + \tfrac{1}{2}(u^2 + v^2 + w^2)$$

Specific enthalpy — Specific total enthalpy

Combining these two definitions with the one for specific energy E

$$h_o = i + p/\rho + \tfrac{1}{2}(u^2 + v^2 + w^2) = E + p/\rho \qquad (2.26)$$

Substituting of (2.26) into (2.22) yields the **(total) enthalpy equation**

$$\frac{\partial(\rho h_o)}{\partial t} + div(\rho h_o \mathbf{u}) = div(k\, grad\, T)$$
$$+\frac{\partial p}{\partial t} + \frac{\partial(u\tau_{xx})}{\partial x} + \frac{\partial(u\tau_{yx})}{\partial y} + \frac{\partial(u\tau_{zx})}{\partial z}$$
$$+\frac{\partial(v\tau_{xy})}{\partial x} + \frac{\partial(v\tau_{yy})}{\partial y} + \frac{\partial(v\tau_{zy})}{\partial z}$$
$$+\frac{\partial(w\tau_{xz})}{\partial x} + \frac{\partial(w\tau_{yz})}{\partial y} + \frac{\partial(w\tau_{zz})}{\partial z} + S_h \qquad (2.27)$$

Equations of State

- Thermodynamic variables: ρ, p, i and T.
- Relationships between the thermodynamic variables can be obtained through the assumption of **thermodynamic equilibrium.**
- **Equations of state** for pressure p and specific internal energy i:

  $p = p(\rho, T)$ and $i = i(\rho, T)$
- For a perfect gas, equations of state are

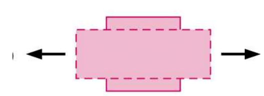
  $p = \rho RT$ and $i = C_v T$
- In the flow of compressible fluids the equations of state provide the linkage between the energy equation and mass conservation and momentum equations.
- Liquids and gases flowing at low speeds behave as incompressible fluids.
- Without density variations there is no linkage between the energy equation and the mass conservation and momentum equations.

Navier-Stokes Equations for a Newtonian Fluid

- We need a suitable model for the viscous stresses τ_{ij}.
- Viscous stresses can be expressed as functions of the local deformation rate (or strain rate).
- In 3D flows the local rate of deformation is composed of the linear deformation rate and the volumetric deformation rate.
- All gases and many liquids are isotropic.
- The rate of linear deformation of a fluid element has nine components in 3D, six of which are independent in isotropic fluids.
- They are denoted by the symbol e_{ij}.

There are three linear elongating deformation components:

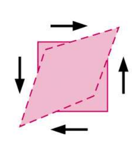
$$e_{xx} = \frac{\partial u}{\partial x} \qquad e_{yy} = \frac{\partial v}{\partial y} \qquad e_{zz} = \frac{\partial w}{\partial z}$$



There are also shearing linear deformation components:

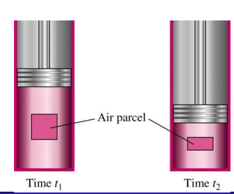
$$e_{xy} = e_{yx} = \tfrac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) \qquad e_{xz} = e_{zx} = \tfrac{1}{2}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)$$

$$e_{yz} = e_{zy} = \tfrac{1}{2}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)$$



The volumetric deformation is given by

$$\frac{1}{V}\frac{DV}{Dt} = \frac{1}{V}\frac{dV}{dt} = \varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}$$



Strain Rate Tensor

We can combine linear strain rate and shear strain rate into one symmetric second-order tensor called the **strain-rate tensor.**

$$\varepsilon_{ij} = \begin{pmatrix} \varepsilon_{xx} & \varepsilon_{xy} & \varepsilon_{xz} \\ \varepsilon_{yx} & \varepsilon_{yy} & \varepsilon_{yz} \\ \varepsilon_{zx} & \varepsilon_{zy} & \varepsilon_{zz} \end{pmatrix} = \begin{pmatrix} \dfrac{\partial u}{\partial x} & \dfrac{1}{2}\left(\dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x}\right) & \dfrac{1}{2}\left(\dfrac{\partial u}{\partial z} + \dfrac{\partial w}{\partial x}\right) \\ \dfrac{1}{2}\left(\dfrac{\partial v}{\partial x} + \dfrac{\partial u}{\partial y}\right) & \dfrac{\partial v}{\partial y} & \dfrac{1}{2}\left(\dfrac{\partial v}{\partial z} + \dfrac{\partial w}{\partial y}\right) \\ \dfrac{1}{2}\left(\dfrac{\partial w}{\partial x} + \dfrac{\partial u}{\partial z}\right) & \dfrac{1}{2}\left(\dfrac{\partial w}{\partial y} + \dfrac{\partial v}{\partial z}\right) & \dfrac{\partial w}{\partial z} \end{pmatrix}$$

- In a Newtonian fluid the viscous stresses are proportional to the rates of deformation.
- The 3D form of Newton's law of viscosity for compressible flows involves two constants of proportionality:

  - The (first) dynamic viscosity, μ, to relate stresses to linear deformations, $(\tau_{ij})_1 = 2\mu\varepsilon_{ij}$

  - The second viscosity, λ, to relate stresses to the volumetric deformation;

$$(\tau_{xx})_2 = (\tau_{yy})_2 = (\tau_{zz})_2 = \lambda(e_{xx} + e_{yy} + e_{yy})$$

The nine viscous stress components, of which six are independent, are

$$\tau_{xx} = 2\mu\frac{\partial u}{\partial x} + \lambda\, div\,\mathbf{u} \qquad \tau_{yy} = 2\mu\frac{\partial v}{\partial y} + \lambda\, div\,\mathbf{u} \qquad \tau_{zz} = 2\mu\frac{\partial w}{\partial z} + \lambda\, div\,\mathbf{u}$$

$$\tau_{xy} = \tau_{yx} = \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) \qquad \tau_{xz} = \tau_{zx} = \mu\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) \qquad (2.31)$$

$$\tau_{yz} = \tau_{zy} = \mu\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)$$

Not much is known about the second viscosity λ, because its effect is small.

For gases a good working approximation is $\lambda = -\frac{2}{3}\mu$

Liquids are incompressible so the mass conservation equation is

$$div\,\mathbf{u} = 0$$

Substitution of the above shear stresses (2.31) into (2.14a-c) yields the Navier-Stokes equations

$$\rho\frac{Du}{Dt} = -\frac{\partial p}{\partial x} + \frac{\partial}{\partial x}\left[2\mu\frac{\partial u}{\partial x} + \lambda div\,\mathbf{u}\right] + \frac{\partial}{\partial y}\left[\mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\right] + \frac{\partial}{\partial z}\left[\mu\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial x}\right)\right] + S_{Mx} \qquad (2.32a)$$

$$\rho\frac{Dv}{Dt} = -\frac{\partial p}{\partial y} + \frac{\partial}{\partial x}\left[\mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\right] + \frac{\partial}{\partial y}\left[2\mu\frac{\partial v}{\partial y} + \lambda div\,\mathbf{u}\right] + \frac{\partial}{\partial z}\left[\mu\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)\right] + S_{My} \qquad (2.32b)$$

$$\rho\frac{Dw}{Dt} = -\frac{\partial p}{\partial z} + \frac{\partial}{\partial x}\left[\mu\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)\right] + \frac{\partial}{\partial y}\left[\mu\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)\right] + \frac{\partial}{\partial z}\left[2\mu\frac{\partial w}{\partial z} + \lambda div\,\mathbf{u}\right] + S_{Mz} \qquad (2.32c)$$

Often it is useful to rearrange the viscous stress terms as follows:

$$\frac{\partial}{\partial x}\left[2\mu\frac{\partial u}{\partial x}+\lambda\, div\,\mathbf{u}\right]+\frac{\partial}{\partial y}\left[\mu\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)\right]+\frac{\partial}{\partial z}\left[\mu\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)\right]$$

$$=\boxed{\frac{\partial}{\partial x}\left(\mu\frac{\partial u}{\partial x}\right)+\frac{\partial}{\partial y}\left(\mu\frac{\partial u}{\partial y}\right)+\frac{\partial}{\partial z}\left(\mu\frac{\partial u}{\partial z}\right)} \longrightarrow div(\mu\, grad\, u)$$

$$+\boxed{\left[\frac{\partial}{\partial x}\left(\mu\frac{\partial u}{\partial x}\right)+\frac{\partial}{\partial y}\left(\mu\frac{\partial v}{\partial x}\right)+\frac{\partial}{\partial z}\left(\mu\frac{\partial w}{\partial x}\right)\right]+\frac{\partial}{\partial x}(\lambda\, div\,\mathbf{u})} \longrightarrow s_{Mx}$$

$$= div(\mu\, grad\, u)+s_{Mx}$$

The viscous stresses in the y- and z-momentum equations can be re-cast in a similar manner.

To simplify the momentum equations:

‘hide’ the smaller contributions to the viscous stress terms in the momentum source.

Defining a new source by

$$S_M = S_M + s_M \qquad (2.33)$$

the Navier-Stokes equations can be written in the most useful form for the development of the finite volume method:

$$\rho\frac{Du}{Dt}=-\frac{\partial p}{\partial x}+div(\mu\, grad\, u)+S_{Mx} \qquad (2.34a)$$

$$\rho\frac{Dv}{Dt}=-\frac{\partial p}{\partial y}+div(\mu\, grad\, v)+S_{My} \qquad (2.34b)$$

$$\rho\frac{Dw}{Dt}=-\frac{\partial p}{\partial z}+div(\mu\, grad\, w)+S_{Mz} \qquad (2.34c)$$

Note that for incompressible fluids, s_M is zero if μ is constant. For example consider s_{Mx}:

$$s_{Mx}=\left[\frac{\partial}{\partial x}\left(\mu\frac{\partial u}{\partial x}\right)+\frac{\partial}{\partial y}\left(\mu\frac{\partial v}{\partial x}\right)+\frac{\partial}{\partial z}\left(\mu\frac{\partial w}{\partial x}\right)\right]+\cancel{\frac{\partial}{\partial x}(\lambda\, div\,\mathbf{u})}$$

$$=\mu\left[\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right)+\frac{\partial}{\partial y}\left(\frac{\partial v}{\partial x}\right)+\frac{\partial}{\partial z}\left(\frac{\partial w}{\partial x}\right)\right]=\mu\left[\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right)+\frac{\partial}{\partial x}\left(\frac{\partial v}{\partial y}\right)+\frac{\partial}{\partial x}\left(\frac{\partial w}{\partial z}\right)\right]$$

$$=\mu\underbrace{\left[\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)\right]}_{=0 \text{ from continuity equation}}=0$$

If we use the Newtonian model for viscous stresses in the internal energy equation (2.24) we obtain

$$\rho\frac{Di}{Dt} = -p\, div\, \mathbf{u} + div(k\, grad\, T) + \Phi + S_i \qquad (2.35)$$

The dissipation function Φ is

$$\Phi = \mu\left\{ \begin{array}{l} 2\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2 + \left(\frac{\partial w}{\partial z}\right)^2\right] + \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)^2 \\ + \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)^2 \end{array} \right\} + \lambda(div\, \mathbf{u})^2 \qquad (2.36)$$

The dissipation function represents a source of internal energy due to deformation work on the fluid particle.

Conservative Form of the Governing Equations of Fluid Flow

The conservative or divergence form of the time-dependent 3-D flow and energy equations of a compressible Newtonian fluid:

Mass	$\frac{\partial \rho}{\partial t} + div(\rho \mathbf{u}) = 0$	(2.4)
x-momentum	$\frac{\partial(\rho u)}{\partial t} + div(\rho u \mathbf{u}) = -\frac{\partial p}{\partial x} + div(\mu\, grad\, u) + S_{Mx}$	(2.37a)
y-momentum	$\frac{\partial(\rho v)}{\partial t} + div(\rho v \mathbf{u}) = -\frac{\partial p}{\partial y} + div(\mu\, grad\, v) + S_{My}$	(2.37b)
z-momentum	$\frac{\partial(\rho w)}{\partial t} + div(\rho w \mathbf{u}) = -\frac{\partial p}{\partial z} + div(\mu\, grad\, w) + S_{Mz}$	(2.37c)
internal energy	$\frac{\partial(\rho i)}{\partial t} + div(\rho i \mathbf{u}) = -p\, div\, \mathbf{u} + div(k\, grad\, T) + \Phi + S_i$	(2.38c)
equations of state	$p = p(\rho, T)$ and $i = i(\rho, T)$	(2.28)
	e.g. for a perfect gas: $p = \rho RT$ and $i = C_v T$	(2.29)

Table 2.1

A system of seven equations with seven unknowns → this system is mathematically closed.

Differential and Integral Forms of the General Transport Equations

Equations in Table 2.1 can usefully be written in the following form:

$$\frac{\partial(\rho\phi)}{\partial t} + div(\rho\phi\mathbf{u}) = div(\Gamma\, grad\, \phi) + S_\phi \qquad (2.39)$$

$$\begin{pmatrix}\text{Rate of increase}\\ \text{of } \phi \text{ of fluid}\\ \text{element}\end{pmatrix} + \begin{pmatrix}\text{Net rate of flow}\\ \text{of } \phi \text{ out of}\\ \text{fluid element}\end{pmatrix} = \begin{pmatrix}\text{Rate of increase}\\ \text{of } \phi \text{ due to}\\ \text{diffusion}\end{pmatrix} + \begin{pmatrix}\text{Rate of increase}\\ \text{of } \phi \text{ due to}\\ \text{sources}\end{pmatrix}$$

Rate of change term — convective term — diffusive term — source term

Equation (2.39) is used as the starting point in finite volume method.

By setting $\left\{\begin{array}{l}\phi = 1, u, v, w, i\\ \Gamma = 0, \mu, k\\ S_\phi = 0, (S_{M_\phi} - \partial p/\partial x), \ldots,\end{array}\right\} \rightarrow$ we obtain equations in Table 2.1.

Flow Equations in Vectorial Form

The x-, y- and z-momentum equations can be written in matrix form as

$$\frac{\partial}{\partial t}\underbrace{\begin{bmatrix}\rho u\\ \rho v\\ \rho w\end{bmatrix}}_{\rho\mathbf{v}} + \underbrace{\begin{bmatrix}\rho uu & \rho uv & \rho uw\\ \rho vu & \rho vv & \rho vw\\ \rho wu & \rho wv & \rho ww\end{bmatrix}}_{\rho\mathbf{vv}}\begin{bmatrix}\nabla_x\\ \nabla_y\\ \nabla_z\end{bmatrix} = \underbrace{\begin{bmatrix}\mu\frac{\partial u}{\partial x} & \mu\frac{\partial u}{\partial y} & \mu\frac{\partial u}{\partial z}\\ \mu\frac{\partial v}{\partial x} & \mu\frac{\partial v}{\partial y} & \mu\frac{\partial v}{\partial z}\\ \mu\frac{\partial w}{\partial x} & \mu\frac{\partial w}{\partial y} & \mu\frac{\partial w}{\partial z}\end{bmatrix}}_{\mu\nabla\mathbf{v}}\begin{bmatrix}\nabla_x\\ \nabla_y\\ \nabla_z\end{bmatrix}$$

$$+\underbrace{\begin{bmatrix}\mu\frac{\partial u}{\partial x} & \mu\frac{\partial v}{\partial x} & \mu\frac{\partial w}{\partial x}\\ \mu\frac{\partial u}{\partial y} & \mu\frac{\partial v}{\partial y} & \mu\frac{\partial w}{\partial y}\\ \mu\frac{\partial u}{\partial z} & \mu\frac{\partial v}{\partial z} & \mu\frac{\partial w}{\partial z}\end{bmatrix}}_{\mu\nabla\mathbf{v}^T}\begin{bmatrix}\nabla_x\\ \nabla_y\\ \nabla_z\end{bmatrix} - \underbrace{\begin{bmatrix}p & & \\ & p & \\ & & p\end{bmatrix}}_{p\mathbf{I}}\begin{bmatrix}\nabla_x\\ \nabla_y\\ \nabla_z\end{bmatrix} + \underbrace{\begin{bmatrix}f_x\\ f_y\\ f_z\end{bmatrix}}_{\mathbf{f}}$$

Then, the conservation of mass and momentum equations can be written in vectorial form as

$$\nabla \cdot (\rho \mathbf{v}) = 0$$

$$\frac{\partial}{\partial t}(\rho \mathbf{v}) + \nabla \cdot (\rho \mathbf{v}\mathbf{v}) = \nabla \cdot (\mu(\nabla \mathbf{v} + \nabla \mathbf{v}^T)) - \nabla p \mathbf{I} + \mathbf{f}$$

I : unit tensor

$\nabla\mathbf{v}$: gradient of velocity vector **v**.

$\nabla\mathbf{v}$ is a tensor given by

$$\nabla \mathbf{v} = \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{bmatrix}$$

f: body forces per unit volume

Similarly, the product **vv** appearing on the left hand side of momentum equations is a tensor given by

$$\mathbf{vv} = \mathbf{v} \otimes \mathbf{v} = \begin{bmatrix} uu & uv & uw \\ vu & vv & vw \\ wu & wv & ww \end{bmatrix}$$

and is obtained from the definition of the dyadic product of two vectors, **a** and **b**, which can be expressed as matrix product of two matrices, the first being a column matrix and the second a row matrix,

$$\mathbf{a} \otimes \mathbf{b} = \begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix} \begin{bmatrix} b_x & b_y & b_z \end{bmatrix} = \begin{bmatrix} a_x b_x & a_x b_y & a_x b_z \\ a_y b_x & a_y b_y & a_y b_z \\ a_z b_x & a_z b_y & a_z b_z \end{bmatrix}$$

Momentum equations can also be written as

$$\frac{\partial}{\partial t}(\rho \mathbf{v}) + \nabla \cdot (\rho \mathbf{v}\mathbf{v}) = \nabla \cdot (\boldsymbol{\tau} - p\mathbf{I})$$

where, the stress tensor $\boldsymbol{\tau}$ for a Newtonian fluid is given in terms of the deformation rates as

$$\boldsymbol{\tau} = \mu(\nabla \mathbf{v} + \nabla \mathbf{v}^T)$$

Neglecting dissipation effects, the energy equation can be written in terms of temperature T as

$$\frac{\partial}{\partial t}(\rho c_p T) + \nabla \cdot (\rho c_p \mathbf{v} T) = \nabla(k \nabla T)$$

In the finite volume method Eqn. (2.39) is integrated over 3-D control volume yielding

$$\int_{CV} \frac{\partial(\rho\phi)}{\partial t} dV + \int_{CV} div(\rho\phi\mathbf{u})dV = \int_{CV} div(\Gamma\, grad\, \phi)dV + \int_{CV} S_\phi dV \qquad (2.40)$$

For a vector **a** Gauss' divergence theorem states

$$\int_{CV} div\, \mathbf{a} dV = \int_A \mathbf{n} \cdot \mathbf{a} dA \qquad (2.41)$$

Applying Gauss' divergence theorem, equation (2.40) can be written as

$$\frac{\partial}{\partial t}\left(\int_{CV} \rho\phi dV\right) + \int_A \mathbf{n} \cdot (\rho\phi\mathbf{u})dA = \int_A \mathbf{n} \cdot (\Gamma\, grad\, \phi)dA + \int_{CV} S_\phi dV \qquad (2.42)$$

Equation (2.42) can be expressed as follows:

$$\begin{pmatrix}\text{Rate of}\\ \text{increase of } \phi\end{pmatrix} + \begin{pmatrix}\text{Net rate of}\\ \text{decrease of } \phi \text{ due to}\\ \text{convection across}\\ \text{the boundaries}\end{pmatrix} = \begin{pmatrix}\text{Rate of increase}\\ \text{of } \phi \text{ due to}\\ \text{diffusion across}\\ \text{the boundaries}\end{pmatrix} + \begin{pmatrix}\text{Net rate of}\\ \text{creation of } \phi\end{pmatrix}$$

In steady state problems the rate of change term of (2.42) is equal to zero.

$$\int_A \mathbf{n}\cdot(\rho\phi\mathbf{u})dA = \int_A \mathbf{n}\cdot(\Gamma\, grad\, \phi)dA + \int_{CV} S_\phi dV \qquad (2.43)$$

Integrating (2.42) with respect to time

$$\int_{\Delta t}\frac{\partial}{\partial t}\left(\int_{CV}\rho\phi dV\right)dt + \int_{\Delta t}\int_A \mathbf{n}\cdot(\rho\phi\mathbf{u})dAdt$$
$$= \int_{\Delta t}\int_A \mathbf{n}\cdot(\Gamma\, grad\, \phi)dAdt + \int_{\Delta t}\int_{CV} S_\phi dVdt \qquad (2.44)$$

Auxiliary Conditions for Viscous Fluid Flow Equations

Table 2-5 Boundary conditions for compressible viscous flow.

Initial conditions for unsteady flows:

- Everywhere in the solution region ρ, **u** and T must be given at time t = 0

Boundary conditions for unsteady and steady flows:

- On solid walls: $\mathbf{u} = \mathbf{u}_w$ (no-slip condition)
  $T = T_w$ (fixed temperature) or $k\partial T/\partial n = -q_w$ (fixed heat flux)
- On fluid boundaries: inlet: ρ, **u** and T must be known as a function of position
  outlet: $-p + \mu\partial u_n/\partial n = F_n$ and $-p + \mu\partial u_t/\partial n = F_t$ (stress continuity)

Suffices: $\left.\begin{array}{l} n \rightarrow \text{normal direction} \\ t \rightarrow \text{tangential direction}\end{array}\right\}$ to boundary

$F \rightarrow$ given surface stress

For incompressible viscous flows:

Table 2.5 is applicable, except that there are no conditions on the density ρ.

Outflow boundaries:

- High Re flows far from solid objects in an external flow
- Fully developed flow out of a duct.

For these boundaries:

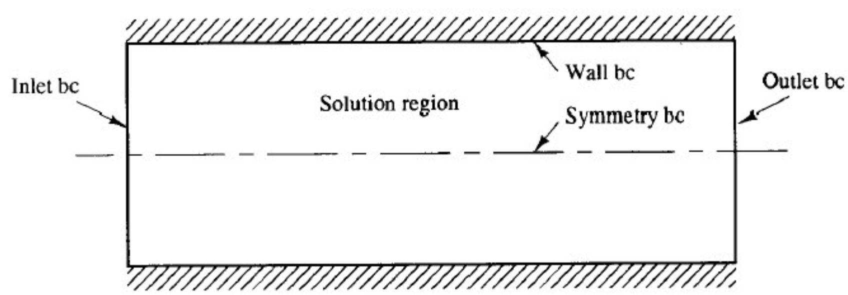
Pressure = specified

$\partial u_n/\partial n = 0$

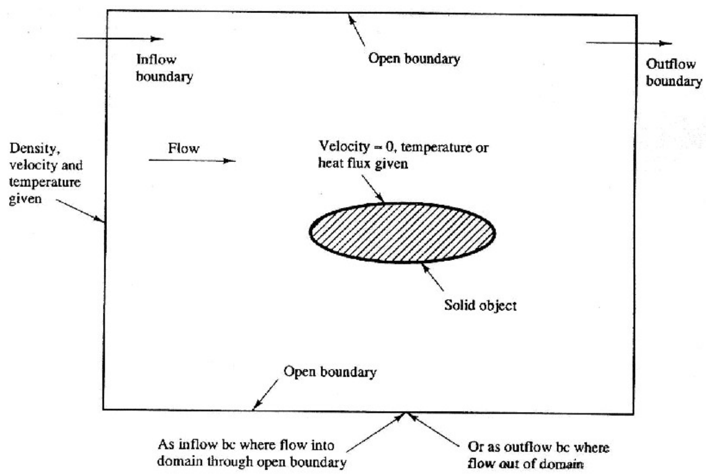
$\partial T/\partial n = 0$

Sources and sinks of mass are placed on the inlet and outlet boundaries to ensure the correct mass flow into and out of domain.

Boundary conditions for an internal flow problem.

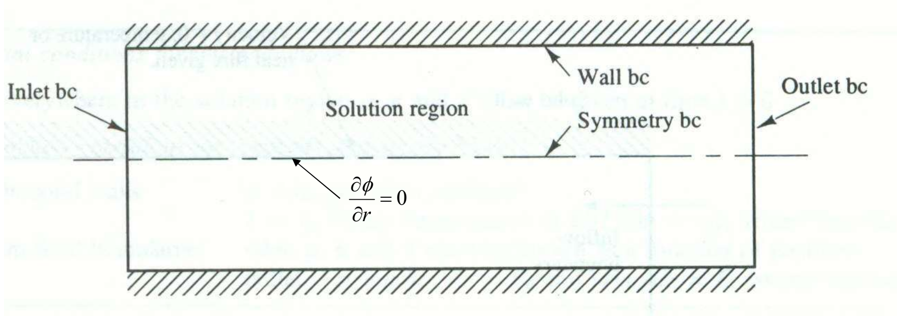


Boundary conditions for an external flow problem.



(b)

Example to symmetry boundary conditions:



Example to cyclic boundary conditions:

Cyclic b.c.: $\phi_1 = \phi_2$