EENG 428 Introduction to Robotics Laboratory

Lab Session 1

Objective

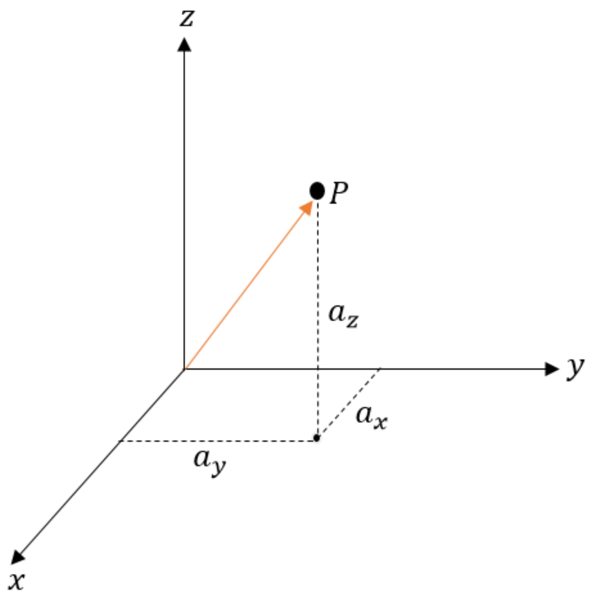
In this Lab session, a review on linear algebra and matrix analysis is presented and described with the assist of the Matlab program to serve the purpose of this course.

1- Point and Vector Representation in Three-Dimensional Space

A point can be represented in space as a vector starting from the origin and ending at the point location using Cartesian coordinates as follow

$$\boldsymbol{P} = \boldsymbol{a_x}\,\hat{\boldsymbol{\imath}} + \boldsymbol{a_y}\,\hat{\boldsymbol{\jmath}} + \boldsymbol{a_z}\,\widehat{\boldsymbol{k}}$$

Where $\hat{\imath}, \hat{\jmath}$ and $\widehat{k}$ are the unit vectors along x, y and z axes respectively. Besides, a_x, a_y and a_z are scalars represent the projections of the point on x, y and z respectively.



An alternative representation for a point in 3-D space using matrices is given by

$$P = [a_x \quad a_y \quad a_z]^T$$

The length (magnitude) of a vector P is defined as

$$\|P\|_2 = \sqrt{a_x{}^2 + a_y{}^2 + a_z{}^2}$$

Any non-zero vector can be normalized to have unity length to represent the direction by dividing each component by the magnitude of the vector.

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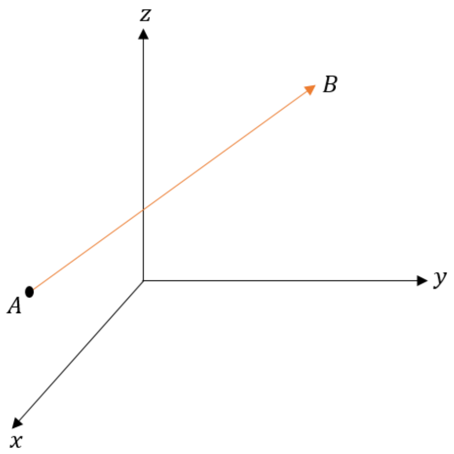
Consider a vector drawn from point A to point B. Point A is called the initial point of the vector, and point B is called the terminal point. Symbolic notation for this vector is $\overrightarrow{AB}$

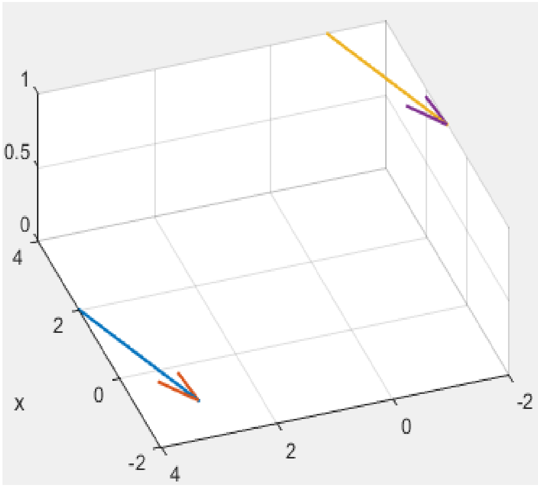
Two vectors are equivalent if they have the same magnitude and direction.



Example 1.1:

The Two Vector A and B Are Equivalent

```
A0=[2 4 0];  A1=[-1 3 0];
B0=[4 -1 1];  B1=[1 -2 1];
vectarrow(A0,A1); hold on;
vectarrow(B0,B1); hold on;
A=[(A1(1)-A0(1)) (A1(2)-A0(2)) (A1(3)-A0(3))]
B=[(B1(1)-B0(1)) (B1(2)-B0(2)) (B1(3)-B0(3))]
%magnitude of A
AM=sqrt((A(1)^2)+(A(2)^2)+(A(3)^2))
%magnitude of B
BM=sqrt((B(1)^2)+(B(2)^2)+(B(3)^2))
%unit vectors
VDirectionA=[A(1)/AM  A(2)/AM  A(3)/AM]
VDirectionB=[B(1)/BM  B(2)/BM B(3)/BM]
```



The representation of vectors can be slightly modified to include a scale factor w such that if x, y and z are divided by w we can reobtain the original vector. In other words, we can represent the vector $\boldsymbol{P} = [\boldsymbol{a_x} \quad \boldsymbol{a_y} \quad \boldsymbol{a_z}]^{\boldsymbol{T}}$ in another form as $\boldsymbol{P} = [\boldsymbol{x} \quad \boldsymbol{y} \quad \boldsymbol{z} \quad \boldsymbol{w}]^{\boldsymbol{T}}$. where $\boldsymbol{a_x} = \boldsymbol{x}/\boldsymbol{w}$, $\boldsymbol{a_y} = \boldsymbol{y}/\boldsymbol{w}$ and $\boldsymbol{a_z} = \boldsymbol{z}/\boldsymbol{w}$ This representation will allow us to generalize a transformation matrix as we will discuss later.

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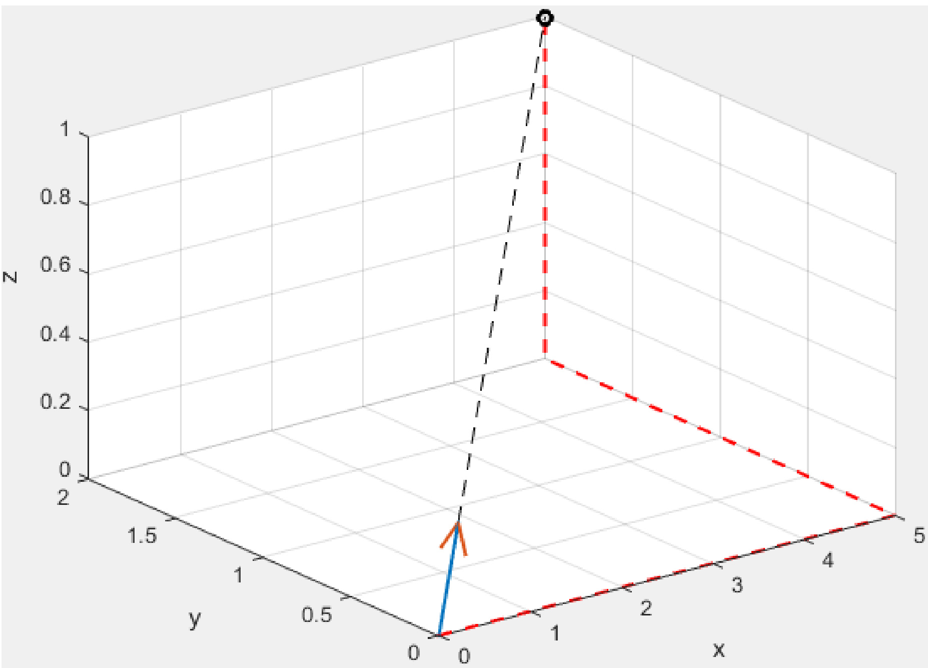
Scale factor w can be any number. However, if $w = 0$, then a_x, a_y and a_z will be undefined. In this case, x, y, and z will represent a vector whose length is undefined, only the direction of vector remains definite. This means that a direction vector can be represented by a scale factor of $w = 0$, where the length is not important.

Example 1.2: Determine the unit directional vector along $P = 5\hat{\imath} + 2\hat{\jmath} + 1\hat{k}$

- The unit directional vector along P is $\hat{P} = \begin{bmatrix} \frac{\sqrt{30}}{6} & \frac{\sqrt{30}}{15} & \frac{\sqrt{30}}{30} \end{bmatrix}^T$

```
x=5;y=2;z=1;
xa=[0 x x x];
ya=[0 0 y y];
za=[0 0 0 z];
plot3(xa,ya,za,'--r','LineWidth',1.5);hold on;
plot3(x,y,z,'kO','LineWidth',2);hold on;grid on;
PM=sqrt(x^2+y^2+z^2) %magnitude of P
p0 = [0 0 0]; p1=[x/PM y/PM z/PM]
vectarrow(p0,p1); hold on;
XC = [x/PM x];
YC = [y/PM y];
ZC = [z/PM z];
plot3(XC,XY,XZ,'--k','LineWidth',0.5);hold on;
```



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2-Vector-Vector Product Operation

2.1- Inner Product (Dot Product)

An operation that takes two vectors and returns a scalar quantity. the dot product between two vectors $A = a_x\,\hat{\imath} + a_y\,\hat{\jmath} + a_z\,\hat{k}$ and $B = b_x\,\hat{\imath} + b_y\,\hat{\jmath} + b_z\,\hat{k}$ can be defined as the sum of the products of the components of each vector as following

$$\boldsymbol{A.B = a_x * b_x + a_y * b_y + a_z * b_z}$$

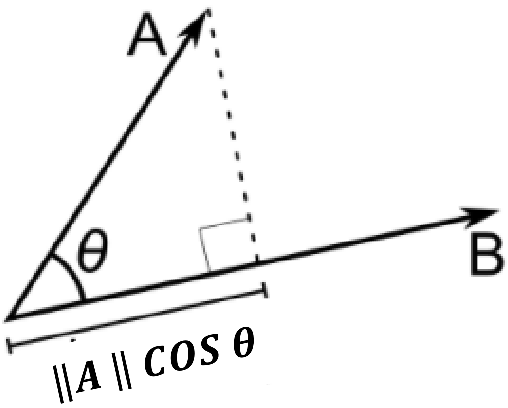
Geometrically, the dot product of two Euclidean Vectors **A** and **B** can be defined as the product of the magnitudes of the two vectors and the cosine of the angle between them. This statement clarified in the following equation

$$\boldsymbol{A.B = \|A\|\ \|B\|\ \cos\theta}$$

Considering the two Vectors A and B. The **scalar projection** (or scalar component) of the Vector A on B is a scalar, equal to the <u>length of the orthogonal projection of A on B</u>.

$$\boldsymbol{A_B = \|A\| \cos\theta}$$

- A_B is the scalar projection of A on B
- θ is the angle between A and B



Properties

- inner product is commutative in real vector subspaces.
- inner product between two non-zero vectors is zero if and only if the vectors are perpendicular (orthogonal).

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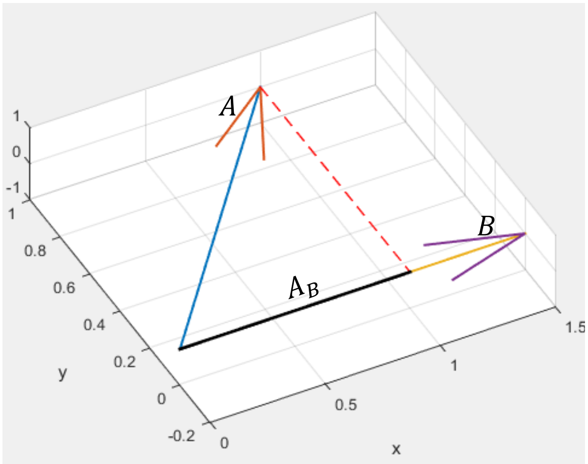
Example 2.1: Consider the two vector $A = 1\hat{\imath} + 1\hat{\jmath} + 0\hat{k}$ and $B = 1.5\hat{\imath} + 0\hat{\jmath} + 0\hat{k}$

Finding the scalar component of the Vector A on B will be as the following

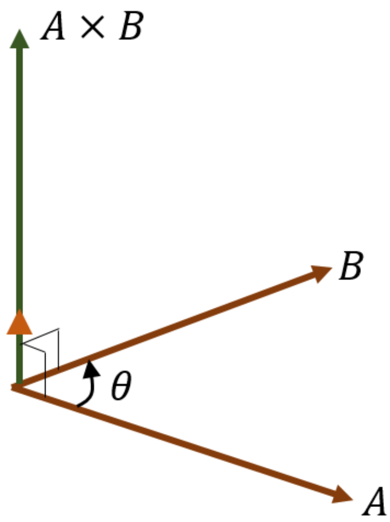
```
p0=[0 0 0];  A=[1 1 0];  B=[1.5
0 0];
vectarrow(p0,A); hold on;
vectarrow(p0,B); hold on;
AM=sqrt(A(1)^2+A(2)^2+A(3)^2);
%magnitude of A
BM=sqrt(B(1)^2+B(2)^2+B(3)^2);
%magnitude of B
ANGLE=dot(A,B)/(AM*BM); %Cosine
of the angle
DA=A/AM; %Direction of A
DB=B/BM;  %Direction of B
PAonB=AM*(ANGLE)*DB;
plot3([A(1) PAonB(1)],[A(2)
PAonB(2)],[A(3) PAonB(3)],'--
r','LineWidth',1);hold on;
plot3([0 PAonB(1)],[0
PAonB(2)],[0 PAonB(3)],'-
k','LineWidth',2)
```



2.2- Outer Product (Cross Product)

The cross product of two vectors A and B is defined only in three-dimensional space as a vector C that is orthogonal to the plan that contain the vectors A and B.

- The direction of the Resultant vector C given by the right-hand rule
- The magnitude of the Resultant vector C equal to the area of the parallelogram that the vectors A and B span.



The cross product is defined by the formula

$$\boldsymbol{A} \times \boldsymbol{B} = \|\boldsymbol{A}\|\|\boldsymbol{B}\| \sin \boldsymbol{\theta}\, \hat{\boldsymbol{n}}$$

$\hat{n}$ is a unit vector orthogonal to the plane formed by A and B

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The cross product could be found algebraically for the vectors $A = a_x\,\hat{\imath} + a_y\,\hat{\jmath} + a_z\,\hat{k}$ and $B = b_x\,\hat{\imath} + b_y\,\hat{\jmath} + b_z\,\hat{k}$ as the normal matrix multiplication of one vector represented in a skew-symmetric matrix by the other

$$\boldsymbol{A} \times \boldsymbol{B} = \begin{bmatrix} \boldsymbol{0} & -\boldsymbol{a_z} & \boldsymbol{a_y} \\ \boldsymbol{a_z} & \boldsymbol{0} & -\boldsymbol{a_x} \\ -\boldsymbol{a_y} & \boldsymbol{a_x} & \boldsymbol{0} \end{bmatrix} * \begin{bmatrix} \boldsymbol{b_x} \\ \boldsymbol{b_y} \\ \boldsymbol{b_z} \end{bmatrix}$$

Properties

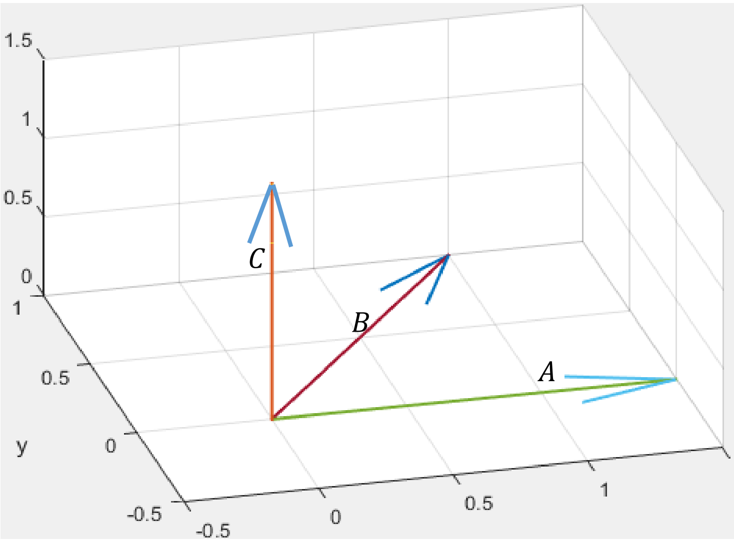
- The cross product is <u>not a commutative</u> operation $A \times B = - B \times A$
- inner product between two non-zero vectors is zero if and only if the vectors are parallel

Example 2.2: Consider the two vectors $A = 1.5\hat{\imath} + 0\hat{\jmath} + 0\hat{k}$ and $B = 1\hat{\imath} + 1\hat{\jmath} + 0\hat{k}$

Finding the scalar component of the Vector A on B will be as the following

```
p0=[0 0 0]; A=[1.5 0 0];
B=[1 1 0];
C1=cross(A,B);
vectarrow(p0,A); hold on;
vectarrow(p0,B); hold on;
vectarrow(p0,C1); hold on;

%C2=cross(B,A);
%vectarrow(p0,C2);hold on;
```



Exercise 1:

Consider the vectors $\boldsymbol{A} = \boldsymbol{1\hat{\imath} + 2\hat{\jmath} + 1\hat{k}}$, $\boldsymbol{B} = \boldsymbol{2\hat{\imath} + 1\hat{\jmath} - 1\hat{k}}$ and $\boldsymbol{C} = \boldsymbol{2\hat{\imath} - 2\hat{\jmath} - 4\hat{k}}$

- Give a scaled representation for each of these vectors using the scalars
- Find a unit vector to represent the direction of a, b and c
- Find the result of $(A \times B).(A - C)$
- Verify your answers using matlab

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3- Moore-Penrose Pseudo Inverse

In mathematics, a system of linear equations (or linear system) is a collection of two or more linear equations involving set of variables. A general system of m linear equations with n unknowns can be written as

$$\begin{aligned} \boldsymbol{a_{11}\, x_1 + a_{12}\, x_2 + \cdots + a_{1n}\, x_n} &= \boldsymbol{b_1} \\ \boldsymbol{a_{21}\, x_1 + a_{22}\, x_2 + \cdots + a_{2n}\, x_n} &= \boldsymbol{b_2} \\ &\vdots \\ \boldsymbol{a_{m1}\, x_1 + a_{m2}\, x_2 + \cdots + a_{mn}\, x_n} &= \boldsymbol{b_m} \end{aligned}$$

Where $x_1, x_2, \ldots\ x_n$ are unknown, $a_{11}, a_{12}, \ldots a_{mn}$ are the coefficients and $b_1, b_1, \ldots b_1$ are the constant terms. The matrix representation form of the system given as

$$\boldsymbol{A\, x = b}$$

Where A is a $m \times n$ matrix, x is a column vector with n entries, and b is a column vector with m entries

$$\boldsymbol{A} = \begin{bmatrix} \boldsymbol{a_{11}} & \boldsymbol{a_{12}} & \cdots & \boldsymbol{a_{1n}} \\ \boldsymbol{a_{21}} & \boldsymbol{a_{22}} & \cdots & \boldsymbol{a_{2n}} \\ \vdots & \vdots & \cdots & \vdots \\ \boldsymbol{a_{m1}} & \boldsymbol{a_{m2}} & \cdots & \boldsymbol{a_{mn}} \end{bmatrix}, \quad \boldsymbol{x} = \begin{bmatrix} \boldsymbol{x_1} \\ \boldsymbol{x_2} \\ \vdots \\ \boldsymbol{x_n} \end{bmatrix} \quad \textbf{and} \quad \boldsymbol{b} = \begin{bmatrix} \boldsymbol{b_1} \\ \boldsymbol{b_2} \\ \vdots \\ \boldsymbol{b_m} \end{bmatrix}$$

A solution of a linear system is an assignment of values to the variables $x_1, x_2, \ldots\ x_n$ such that each of the equations is satisfied. A linear system may behave in any one of three possible ways: The system has **infinitely many solutions**, the system has **a single unique solution** and the system has **no solution**.

In mathematics, a pseudoinverse (Moore–Penrose inverse) A^+ of a matrix A is a generalization of the inverse matrix solutions. It is proved that pseudoinverse A+ of the matrix *A* is unique matrix.

$$\boldsymbol{A^+ = \lim_{\delta \to 0} (A^T A + \delta^2 I)^{-1} A^T = \lim_{\delta \to 0} A^T (A\, A^T + \delta^2 I)^{-1}}$$

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Moore and Penrose showed that there is a general solution (Moore-Penrose solution) to the system of equations $\boldsymbol{A}\,\boldsymbol{x} = \boldsymbol{b}$ of the form $\boldsymbol{x} = \boldsymbol{A}^{+}\,\boldsymbol{b}$

Considering the matrix $A \in R_r^{m\times n}$

1- If $\boldsymbol{m} > \boldsymbol{n}$, in this case there are **more constraining equations than variables** (which corresponds to a **kinematically insufficient manipulator**). Hence, it is **not generally possible to find a solution** to these equations. The pseudoinverse gives the solution that **minimizes** the quantity $\|\boldsymbol{b} - \boldsymbol{A}\boldsymbol{x}\|$

Example 3.1

Determine the **General** solution (Moore-Penrose solution) to the system of equations illustrated below

$$\begin{aligned} \boldsymbol{x_1} + \boldsymbol{x_2} &= \boldsymbol{1} \\ \boldsymbol{2x_1} + \boldsymbol{3x_2} &= \boldsymbol{3} \\ \boldsymbol{7x_1} + \boldsymbol{4x_2} &= \boldsymbol{4} \end{aligned}$$

$$\begin{aligned} \boldsymbol{x_1} + \boldsymbol{x_2} &= \boldsymbol{1} \\ \boldsymbol{2x_1} + \boldsymbol{3x_2} &= \boldsymbol{3} \\ \boldsymbol{7x_1} + \boldsymbol{4x_2} &= \boldsymbol{2} \end{aligned}$$

```
clear all
A=[1 1 ;2 3; 7 4]
Aplus=pinv(A) %A+
x=Aplus*[1;3;4] %solution
E1=1*x(1)+ 1*x(2)
E2=2*x(1)+ 3*x(2)
E3=7*x(1)+ 4*x(2)
```

```
>> Aplus =
  -0.0503  -0.2961   0.2346
   0.1061   0.5140  -0.1620
>> x =
   0.0        1.00
>> E1 = 1    E2 = 3    E3 = 4
```

```
clear all
A=[1 1 ;2 3; 7 4]
Aplus=pinv(A) %A+
x=Aplus*[1;3;2] %solution
E1=1*x(1)+ 1*x(2)
E2=2*x(1)+ 3*x(2)
E3=7*x(1)+ 4*x(2)
```

```
>> Aplus =
  -0.0503  -0.2961   0.2346
   0.1061   0.5140  -0.1620
>> x =
   -0.4693   1.3240
>> E1 = 0.8547    E2 = 3.0335    E3 = 2.0112
```


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2- If $m < n$ (which corresponds to a **<u>kinematically redundant manipulator</u>**). In this case, there **<u>may be</u> more than one possible solution** (generally an infinite number of solutions). The Moore-Penrose solution is the particular solution whose vector 2-norm $\|x\|$ is minimal

Example 3.2

Determine the **<u>General</u>** solution (Moore-Penrose solution) to the system of equations illustrated below

$$x_1 + 2x_2 + x3 = 2$$
$$2x_1 + 2x_2 + 3x3 = 3$$

```
clear all
A=[1 2 1 ;2 2 3]
Aplus=pinv(A) %A+
x=Aplus*[2;3] % the solution
E1=1*x(1)+ 2*x(2)+ 1*x(3)
E2=2*x(1)+ 2*x(2)+ 3*x(3)
Vnorm=norm(x,2)
```

```
>> Aplus =
  -0.0476   0.1429
   0.7619  -0.2857
  -0.4762   0.4286
>> x =
0.3333  0.6667  0.3333
 >> E1 = 2   E2 = 3
Vnorm = 0.8165
```

Even though $x = [1\ 0.5\ 0]$ whose vector 2-norm is 1.118 represent a solution to the given system, but the this solution is not **<u>General solution</u>** that is the particular solution whose 2-norm $\|x\|$ is minimal

3- If $m = n$ and the matrix A is invertible (**square, full rank matrix**) then the pseudoinverse identical to the normal inverse $A^{+} = A^{-1}$. Even if the matrix A is not invertible (square but not full rank) in this case the determinant of the matrix A is zero subsequently there is no possible way to find the inversion of the matrix A in the normal way. The More-Penrose pseudoinverse can be used in such cases.

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Properties

- If a matrix A is not-invertible, but $A^T A$ is invertible, the pseudo-inverse will be easy to find (respectively), and is called as the left inverse

$$\boldsymbol{A^+ = (A^T A)^{-1} A^T}$$

- If a matrix A is not-invertible, but $A\,A^T$ is invertible, the pseudo-inverse will be easy to find (respectively), and is called as the right inverse.

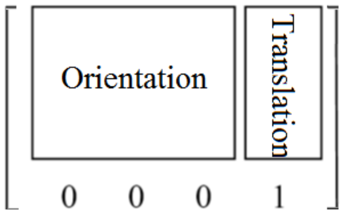
$$\boldsymbol{A^+ = A^T (A\,A^T)^{-1}}$$

Exercise 2:

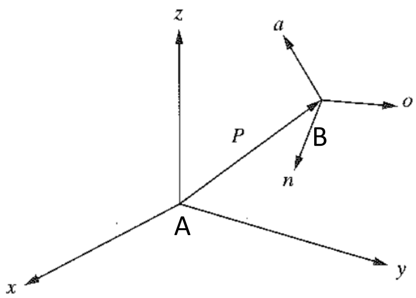
Find by hand the pseudo-inverse of the Matrix $A = \begin{bmatrix} 1 & 1 \\ 3 & 3 \end{bmatrix}$

4- General Transformation Matrices

A general transformation is a Block matrix that combines two sets of information, the orientation of one coordinate frame (the non-reference frame) w.r.t another (the reference frame), and the position of the non-reference frame w.r.t the reference frame



In other words, a frame can be expressed in terms of the fixed reference frame by three vectors describing its directional unit vector, as well as the forth vector describing its location



$$^{A}T_{B} = \begin{bmatrix} n_x & o_x & a_x & P_x \\ n_y & o_y & a_y & P_y \\ n_z & o_z & a_z & P_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

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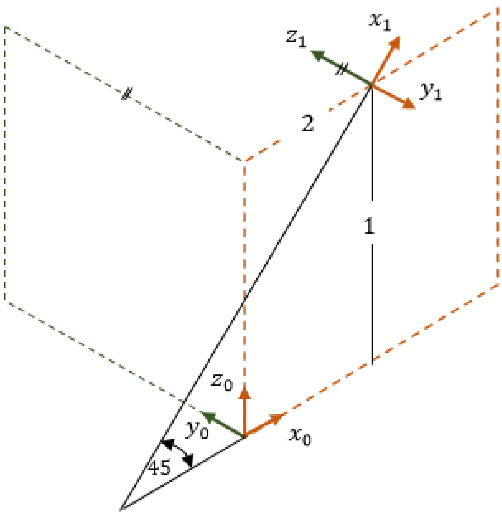
Example:

Finding the 4 by 4 homogeneous transformation matrix which represent the frame 1 relative to the frame 0

$$^{0}T_1 = \begin{bmatrix} \cos 45 & \cos 45 & 0 & 2 \\ 0 & 0 & 1 & 0 \\ \sin 45 & -\sin 45 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$




```
%% General Transformation Matrix
To=[1/sqrt(2)     1/sqrt(2)      0       2
          0           0          1       1
    1/sqrt(2)    -1/sqrt(2)      0       1
          0           0          0       1];
%% Representative tools to draw the relation btw
two Frames
Hfr = [0 1 0 0 0 0 0
       0 0 0 1 0 0 0
       0 0 0 0 0 1 0
       1 1 1 1 1 1 1 ];
frame0={'';'x0';'';'y0';'';'z0';''};
frame1={'';'x1';'';'y1';'';'z1';''};
xp=To(1,4) ; yp=To(2,4); zp = To(3,4);
plot3( Hfr(1,:), Hfr(2,:), Hfr(3,:) ,'b',
'LineWidth',2);
hold on;
xlabel('X');ylabel('Y');zlabel('Z');
S1=size(Hfr);
for i=1:S1(2)
    c=Hfr(4,i); if c==0, c=1;end
    x=Hfr(1,i)/c;y=Hfr(2,i)/c;z=Hfr(3,i)/c;
    text(x,y,z,frame0(i));
end
Tfr = To * Hfr;
plot3( Tfr(1,:), Tfr(2,:), Tfr(3,:) ,'r-.',
'LineWidth',2);
axis equal; grid on;
S2=size(Tfr);
for i=1:S2(2)
    c=Tfr(4,i); if c==0, c=1;end
    x=Tfr(1,i)/c;y=Tfr(2,i)/c;z=Tfr(3,i)/c;
    text(x,y,z,frame1(i));
end
axis equal;
hold on;
x=xp; y=yp; z=zp;
xa= [ 0 x x x];ya= [ 0 0 y y];za= [ 0 0 0 z];
plot3(xa, ya, za);hold on;
plot3( x, y, z, 'ko', 'LineWidth',2);
```

Important Properties

Orthogonal matrices

A matrix is said to be orthogonal if and only if

1- All its columns (rows) are unit vectors (vectors of length 1).

2- Each of its columns (rows) is perpendicular to the rest.

If A is an orthogonal matrix, then

$$\begin{cases} A^T A = A\,A^T = I_n \\ A^T = A^{-1} \\ Det(A) = 1 \end{cases}$$

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It is proven that the **Orientation Part** of the General Transformation matrix is **Orthogonal matrix** based on the definition of the General transformation matrix (check [3] for prove)

Inversion of the General Transformation Matrix

The General Transformation Matrix was introduced as a block-type matrix from the form:

$$^{A}T_{B} = \begin{bmatrix} R & P \\ 0 & 1 \end{bmatrix}$$

Where R is an orthogonal matrix, and P is a vector.

This matrix is similar to the block matrix $\begin{bmatrix} A & B \\ 0 & D \end{bmatrix}$ which is known to have the following property

$$\begin{bmatrix} A & B \\ 0 & D \end{bmatrix}^{-1} = \begin{bmatrix} A^{-1} & -A^{-1}B\,D^{-1} \\ 0 & D^{-1} \end{bmatrix}$$

Then, the inverse of the general transformation matrix $^{A}T_{B}$ as follow

$$\begin{bmatrix} \boldsymbol{R} & \boldsymbol{P} \\ \boldsymbol{0} & \boldsymbol{1} \end{bmatrix}^{-1} = \begin{bmatrix} \boldsymbol{R}^{\boldsymbol{T}} & -\boldsymbol{R}^{\boldsymbol{T}}\boldsymbol{P} \\ \boldsymbol{0} & \boldsymbol{1} \end{bmatrix}$$

- ✓ If $^{A}T_{B}$ represents the frame B w.r.t the frame A then,

$$^{B}T_{A} = \left(^{A}T_{B} \right)^{-1}$$

References:

1- Niku, S. (2010). *Introduction to robotics*. John Wiley & Sons.

2- Laub, A. J. (2005). *Matrix analysis for scientists and engineers*(Vol. 91). Siam.

3- Rotation Matrix (2019). In Wikipedia. Retrieved from https://en.wikipedia.org/wiki/Rotation_matrix

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