Chapter 4:

The Finite Volume Method for Diffusion Problems

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Introduction

General transport equation is

$$\frac{\partial(\rho\phi)}{\partial t} + div(\rho\phi u) = div(\Gamma grad\phi) + S_\phi$$

For steady diffusion:

$$div(\Gamma grad\phi) + S_\phi = 0$$

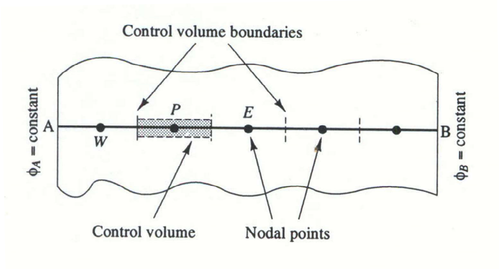
Control volume integration gives

$$\int_{CV} div(\Gamma grad\phi)dV + \int_{CV} S_\phi dV = \int_A \mathbf{n}\cdot(\Gamma grad\phi)dA + \int_{CV} S_\phi dV$$

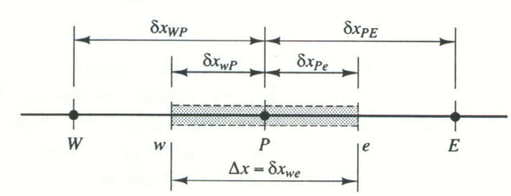
Finite Volume Method for One-dimensional Steady State Diffusion

Steady-state diffusion of a general property ϕ in one-dimensional domain is

$$\frac{d}{dx}(\Gamma\frac{d\phi}{dx}) + S_\phi = 0$$



Step 1: Grid generation:



Step 2: Discretisation

Integration of the diffusion equation over the CV gives

$$\int_{\Delta V}\frac{d}{dx}\left(\Gamma\frac{d\phi}{dx}\right)dV + \int_{\Delta V} S_\phi dV = \left(\Gamma A\frac{d\phi}{dx}\right)_e - \left(\Gamma A\frac{d\phi}{dx}\right)_w + \overline{S}\Delta V = 0$$

To find expressions at the east and west faces, use Taylor series approximations

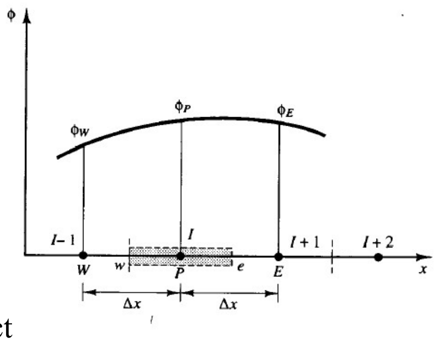
$$\phi(x+\Delta x) = \phi(x) + \left(\frac{\partial\phi}{\partial x}\right)_x \Delta x + \left(\frac{\partial^2\phi}{\partial x^2}\right)_x \frac{\Delta x^2}{2} + \cdots$$

$$\phi(x+\Delta x) = \phi(x) + \left(\frac{\partial \phi}{\partial x}\right)_x \Delta x + \left(\frac{\partial^2 \phi}{\partial x^2}\right)_x \frac{\Delta x^2}{2} + \cdots$$

$$\phi_E = \phi_P + \left(\frac{\partial \phi}{\partial x}\right)_P \Delta x + \boxed{\left(\frac{\partial^2 \phi}{\partial x^2}\right)_P \frac{\Delta x^2}{2} + \cdots}$$

$$\phi_W = \phi_P - \left(\frac{\partial \phi}{\partial x}\right)_P \Delta x + \boxed{\left(\frac{\partial^2 \phi}{\partial x^2}\right)_P \frac{\Delta x^2}{2} + \cdots}$$

Neglect



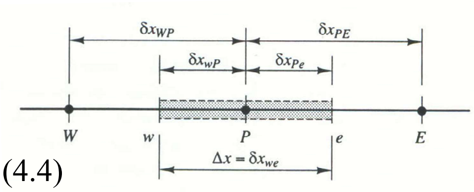
Adding and subtracting → $\left(\frac{\partial \phi}{\partial x}\right)_P = \frac{\phi_E - \phi_W}{2\Delta x}$

At the east face → $\left(\frac{\partial \phi}{\partial x}\right)_e = \frac{\phi_E - \phi_P}{\Delta x}$

Rewriting the diffusion equation for an interior point P:

$$\int_{\Delta V} \frac{d}{dx}\left(\Gamma \frac{d\phi}{dx}\right) dV + \int_{\Delta V} S_\phi dV = 0$$

$$\left(\Gamma A \frac{d\phi}{dx}\right)_e - \left(\Gamma A \frac{d\phi}{dx}\right)_w + \bar{S}_\phi \Delta V = 0 \qquad (4.4)$$



On a uniform grid linear interpolation of Γ is

$$\Gamma_w = \frac{\Gamma_W + \Gamma_P}{2} \qquad \Gamma_e = \frac{\Gamma_P + \Gamma_E}{2} \qquad (4.5)$$

Diffusive flux terms are

$$\left(\Gamma A \frac{d\phi}{dx}\right)_e = \Gamma_e A_e \left(\frac{\phi_E - \phi_P}{\delta x_{PE}}\right) \qquad (4.6)$$

$$\left(\Gamma A \frac{d\phi}{dx}\right)_w = \Gamma_w A_w \left(\frac{\phi_P - \phi_W}{\delta x_{WP}}\right) \qquad (4.7)$$

The source term S may be a function of ϕ → express S in linear form as:

$$S_\phi = S_u + S_p \phi_P \qquad (4.8)$$

Substituting (4.6), (4.7) and (4.8) into (4.4)

$$-\Gamma_e A_e \left(\frac{\phi_E - \phi_P}{\delta x_{PE}} \right) + \Gamma_w A_w \left(\frac{\phi_P - \phi_W}{\delta x_{WP}} \right) = (S_u + S_p \phi_P) \Delta V \qquad (4.9)$$

Rearranging,

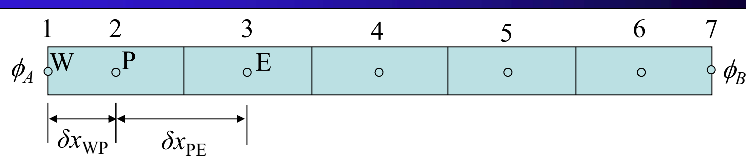
$$\left(\frac{\Gamma_e}{\delta x_{PE}} A_e + \frac{\Gamma_w}{\delta x_{WP}} A_w - S_p \Delta V \right) \phi_P - \left(\frac{\Gamma_w}{\delta x_{WP}} A_w \right) \phi_W - \left(\frac{\Gamma_e}{\delta x_{PE}} A_e \right) \phi_E = S_u \Delta V \qquad (4.10)$$

or,

$$a_P \phi_P + a_W \phi_W + a_E \phi_E = S \qquad (4.11)$$

where,

a_W	a_E	a_P	S
$-\dfrac{\Gamma_w A_w}{\delta x_{WP}}$	$-\dfrac{\Gamma_e A_e}{\delta x_{PE}}$	$-a_W - a_E - S_p \Delta V$	$S_u \Delta V$



Special case: no source terms ($S_\phi = 0$), boundary values ϕ_A , ϕ_B specified.

For the point near a west boundary (point 2):

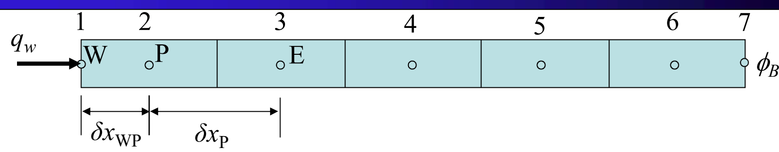
$$\Gamma_e A_e \left(\frac{\phi_E - \phi_P}{\delta x_{PE}} \right) - \Gamma_w A_w \left(\frac{\phi_P - \phi_A}{\delta x_{WP}} \right) = 0$$

$$\left(\frac{\Gamma_e}{\delta x_{PE}} A_e + \frac{\Gamma_w}{\delta x_{WP}} A_w \right) \phi_P - 0.\phi_W - \left(\frac{\Gamma_e}{\delta x_{PE}} A_e \right) \phi_E = \left(\frac{\Gamma_w}{\delta x_{WP}} A_w \right) \phi_A$$

or,

$$a_P \phi_P + a_W \phi_W + a_E \phi_E = S$$

a_W	a_E	a_P		S
0	$-\dfrac{\Gamma_e A_e}{\delta x_{PE}}$	$-(a_W + a_E)_{\text{int}}$		$(a_W)_{\text{int}} \phi_A$



No source terms ($S_\phi = 0$), heat flux q_w specified at west boundary.

For the point near a west boundary (point 2):

$$\Gamma_e A_e \left(\frac{\phi_E - \phi_P}{\delta x_{PE}} \right) + q_w A_w = 0$$

$$\left(\frac{\Gamma_e}{\delta x_{PE}} A_e \right) \phi_P = 0.\phi_W + \left(\frac{\Gamma_e}{\delta x_{PE}} A_e \right) \phi_E + q_w A_w$$

or,

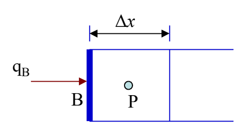
$$a_P \phi_P = a_W \phi_W + a_E \phi_E + S_u$$

a_W	a_E	a_P	S_P	S_u
0	$\frac{\Gamma_e A_e}{\delta x_{PE}}$	$a_W + a_E - S_p$	0	$q_w A_w$

Summary of Boundary Conditions

For a one-dimensional CV of width Δx near boundary B:

1) Set coefficient $a_B(i) = 0$ $(i \rightarrow P)$



2) Add source contributions

(a) Fixed value ϕ_B:

Add: $$S_u = \frac{k_B A_B}{\Delta x / 2} \phi_B$$

$$S_p = -\frac{k_B A_B}{\Delta x / 2}$$

to the source terms S_u and S_p

(b) Fixed flux q_B:

Add $q_B A_B$ in the form of $S_u + S_p \phi_P$ to the source terms S_u and S_p.

Step 3: Solution of equations

Discretised equations of the form (4.11)

$$a_P\phi_P = a_W\phi_W + a_E\phi_E + S_u \qquad (4\text{-}11)$$

must be set up at each nodal point.

→We obtain a system of linear algebraic equations

Solve the system for ϕ values

→Use any matrix solution method.

e.g. Tri-diagonal matrix algorithm (see textbook)

or: Gauss Seidel iteration method.

Gauss Seidel Iteration Method

For a 2-D problem consider the equation for point P in the form

$$a_P\phi_P = a_W\phi_W + a_E\phi_E + a_S\phi_S + a_N\phi_N + S_u$$

ϕ_P at iteration n+1 is calculated from

$$\phi_P^{n+1} = \frac{1}{a_P}\left(a_W\phi_W^n + a_E\phi_E^n + a_S\phi_S^n + a_N\phi_N^n + S_u\right)$$

where superscript n refers to the latest available values; ie some are at the previous iteration n and some at n+1.

Iterations are repeated until the changes in all ϕ_P values fall below a prescribed tolerance.

For one dimensional problems the iteration equation is

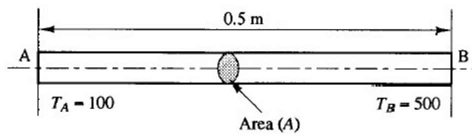
$$\phi_P^{n+1} = \frac{1}{a_P}\left(a_W\phi_W^n + a_E\phi_E^n + S_u\right)$$

Worked Examples: 1) One-dimensional steady state diffusion

Governing equation $\rightarrow \dfrac{d}{dx}\left(k\dfrac{dT}{dx}\right)+S=0$

$k \rightarrow \Gamma$, $\phi \rightarrow T$, S = heat generation per unit volume

Consider the problem of source-free heat conduction in an insulated rod whose ends are maintained at constant temperatures of 100 ^{o}C and 500 ^{o}C respectively. Calculate the steady state temperature in the rod. Take thermal conductivity k = 1000 W/mK, cross-sectional area A = 10×10^{-3}



In this case, S = 0

Solution: Let us divide the rod into 5 equal control volumes (CV's).

Rules for grid generation:

1) Locations of the CV faces are defined first.

2) Then nodal points are placed at the centers of the CV's.

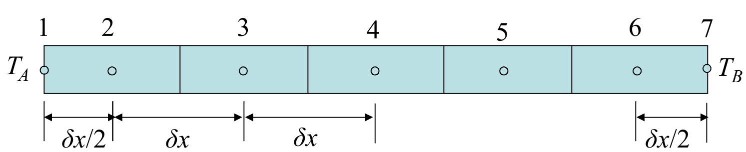
3) Numbering starts from the boundary node at left.

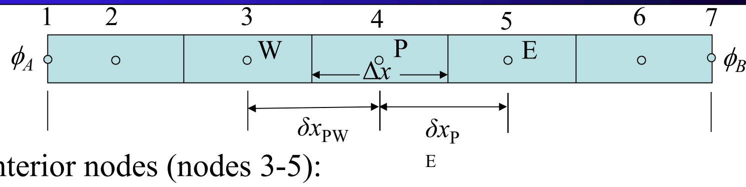
4) All CV's have a volume of $\delta x.A$

5) Inter-nodal distances are equal to δx, ($\delta x_{WP} = \delta x_{PE} = \delta x$)

6) Near west boundary (node 2), $\delta x_{WP} = \delta x/2$

7) Near east boundary (node 6), $\delta x_{PE} = \delta x/2$





(4.8)

For interior nodes (nodes 3-5): (4.9)

$$\Gamma_e A_e \left(\frac{\phi_E - \phi_P}{\delta x_{PE}}\right) - \Gamma_w A_w \left(\frac{\phi_P - \phi_W}{\delta x_{WP}}\right) = 0$$

$$\left(\frac{\Gamma_e}{\delta x_{PE}} A_e + \frac{\Gamma_w}{\delta x_{WP}} A_w\right) T_P = \left(\frac{\Gamma_w}{\delta x_{WP}} A_w\right) T_W + \left(\frac{\Gamma_e}{\delta x_{PE}} A_e\right) T_E \quad (4.10)$$

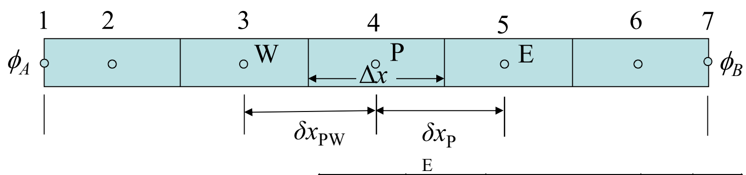
or, $$a_P T_P = a_W T_W + a_E T_E \quad (4.11)$$

where,

a_W	a_E	a_P
$\frac{\Gamma_w A_w}{\delta x_{WP}}$	$\frac{\Gamma_e A_e}{\delta x_{PE}}$	$a_W + a_E - S_p$

(4-11)

General equation: $a_P \phi_P = a_W \phi_W + a_E \phi_E + S_u$



Interior nodes (nodes 3-5):
$S_u = 0,\ S_p = 0,\ \Gamma = k$

a_W	a_E	a_P	S_p	S_u
$\frac{\Gamma_w A_w}{\delta x_{WP}}$	$\frac{\Gamma_e A_e}{\delta x_{PE}}$	$a_W + a_E - S_p$	0	0

For boundary node 2:

a_W	a_E	a_P	S_p	S_u
0	$\frac{\Gamma_e A_e}{\delta x_{PE}}$	$a_W + a_E - S_p$	$-\frac{kA}{\delta x_{WP}}$	$\frac{kA}{\delta x_{WP}} T_A$

For boundary node 6:

a_W	a_E	a_P	S_p	S_u
$\frac{\Gamma_w A_w}{\delta x_{WP}}$	0	$a_W + a_E - S_p$	$-\frac{kA}{\delta x_{PE}}$	$\frac{kA}{\delta x_{PE}} T_B$

The resulting system of equations are

$$\begin{bmatrix} -a_{P_2} & a_{E_2} & & & & & & & \\ a_{W_3} & -a_{p_3} & a_{E_3} & & & & & & \\ & a_{W_4} & -a_{p_4} & a_{E_4} & & & & & \\ & & & & & & & & \\ & & & a_{W_i} & -a_{p_i} & a_{E_i} & & & \\ & & & & & & & & \\ & & & & & a_{W_{n-2}} & -a_{p_{n-2}} & a_{E_{n-2}} \\ & & & & & & a_{W_{n-1}} & -a_{p_{n-1}} \end{bmatrix} \begin{bmatrix} T_2 \\ T_3 \\ T_4 \\ \\ T_i \\ \\ T_{n-2} \\ T_{n-1} \end{bmatrix} = \begin{bmatrix} -Su_2 \\ -Su_3 \\ -Su_4 \\ \\ -Su_i \\ \\ -Su_{n-2} \\ -Su_{n-1} \end{bmatrix}$$

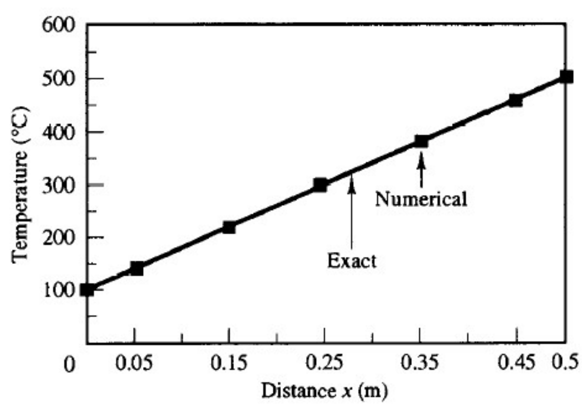
Solve the system of equations using Tri-diagonal matrix algorithm (TDMA) for T_2, T_3, $\ldots T_{n\text{-}1}$, where ($n = 7$). Or use Gauss-Seidel method

The solution is:

$$\begin{bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \\ T_6 \\ T_7 \end{bmatrix} = \begin{bmatrix} 100 \\ 140 \\ 220 \\ 300 \\ 380 \\ 460 \\ 500 \end{bmatrix}$$

Exact solution is:

$$T = 800x + 100$$

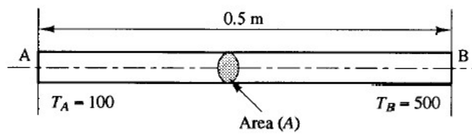


Comparison of the numerical result with the analytical solution.

Homework 1:

Write a computer program to find the temperature distribution in the problem given in Example 1. Use 5 control volumes.

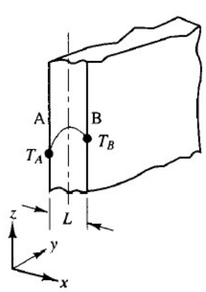
Use the algorithm given in the "Pseudo program to find the a_E coefficients and source terms in 1-D Diffusion Problems" given in page 34 of the lecture notes.



Example 4.2 Now we discuss a problem that includes sources other than those arising from boundary conditions.

Figure 4.6 shows a large plate of thickness $L = 2$ cm with constant thermal conductivity $k = 0.5$ W/m/K and uniform heat generation $q = 1000$ kW/m^3. The faces A and B are at temperatures of 100 °C and 200 °C respectively. Assuming that the dimensions in the y- and z-directions are so large that temperature gradients are significant in the x-direction only, calculate the steady state temperature distribution. Compare the numerical result with the analytical solution. The governing equation is

$$\frac{d}{dx}\left(k\frac{dT}{dx}\right) + q = 0 \qquad (4.25)$$



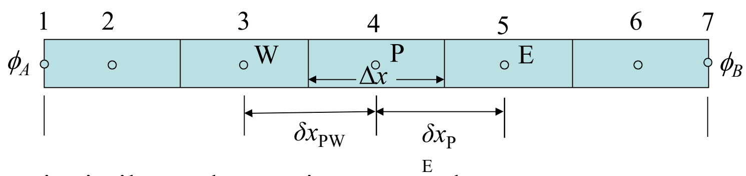
The governing equation is: $\frac{d}{dx}\left(k\frac{dT}{dx}\right)+\dot{q}=0$

The general equation is: $\frac{d}{dx}\left(\Gamma\frac{d\phi}{dx}\right)+S=0$

Comparing the above equations, where $S\Delta V = S_u + S_p\phi_P$

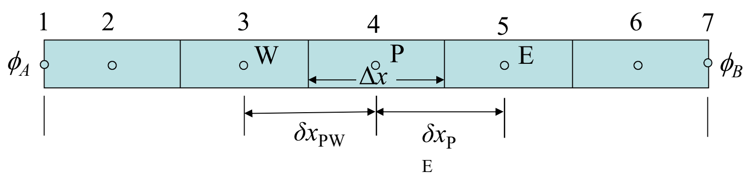
$\phi = T, \Gamma = k, \; S_u = q\Delta V \quad S_p = 0$ where, $\Delta V = A\Delta x$

Take area A = 1 in the y-z plane



Solution is similar to the previous example.

General equation: $a_P\phi_P = a_W\phi_W + a_E\phi_E + S_u$



Interior nodes (nodes 3-5):

$S_u = \dot{q}A\Delta x, \; S_p = 0, \Gamma = k$

a_W	a_E	a_P	S_p	S_u
$\frac{\Gamma_w A_w}{\delta x_{WP}}$	$\frac{\Gamma_e A_e}{\delta x_{PE}}$	$a_W + a_E - S_p$	0	$\dot{q}A\Delta x$

For boundary node 2:

a_W	a_E	a_P	S_p	S_u
0	$\frac{\Gamma_e A_e}{\delta x_{PE}}$	$a_W + a_E - S_p$	$-\frac{k_w A_w}{\delta x_{WP}}$	$\dot{q}A\Delta x + \frac{k_w A_w}{\delta x_{WP}}T_A$

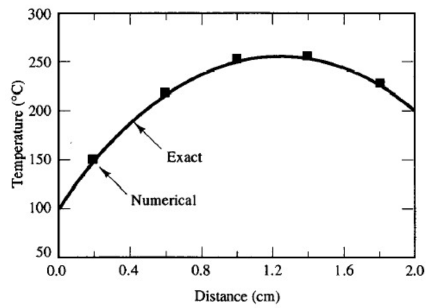
For boundary node 6:

a_W	a_E	a_P	S_p	S_u
$\frac{\Gamma_w A_w}{\delta x_{WP}}$	0	$a_W + a_E - S_p$	$-\frac{k_e A_e}{\delta x_{PE}}$	$\dot{q}A\Delta x + \frac{k_e A_e}{\delta x_{PE}}T_B$

Node number	2	3	4	5	6
x (m)	0.002	0.006	0.01	0.014	0.018
Finite volume solution	150	218	254	258	230
Exact solution	146	214	250	254	226
Percentage error	2.73	1.86	1.60	1.57	1.76

The solution is:

$$\begin{bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \\ T_6 \\ T_7 \end{bmatrix} = \begin{bmatrix} 100 \\ 150 \\ 218 \\ 254 \\ 258 \\ 230 \\ 200 \end{bmatrix}$$



Comparison of the numerical result with the analytical solution.

Exact solution is: $$T = \left[\frac{T_B - T_A}{L} + \frac{q}{2k}(L - x)\right] x + T_A$$

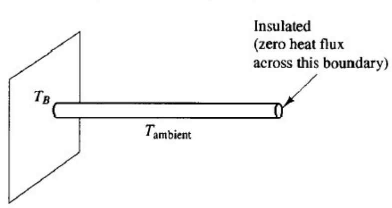
Shown in Figure 4.9 is a cylindrical fin with uniform cross-sectional area A. The base is at a temperature of 100 °C (T_B) and the end is insulated. The fin is exposed to an ambient temperature of 20 °C. One-dimensional heat transfer in this situation is governed by

$$\frac{d}{dx}\left(kA\frac{dT}{dx}\right) - hP(T - T_\infty) = 0 \qquad (4.40)$$

where h is the convective heat transfer coefficient, P the perimeter, k the thermal conductivity of the material and T_∞ the ambient temperature. Calculate the temperature distribution along the fin and compare the results with the analytical solution given by

$$\frac{T - T_\infty}{T_B - T_\infty} = \frac{\cosh[n(L - x)]}{\cosh(nL)} \qquad (4.41)$$

where $n^2 = hP/(kA)$, L is the length of the fin and x the distance along the fin. Data: $L = 1$ m, $hP/(kA) = 25\ m^{-2}$ (note kA is constant).



The governing equation is: $\frac{d}{dx}\left(kA\frac{dT}{dx}\right)-hP(T-T_\infty)=0$ or $\frac{d}{dx}\left(\frac{dT}{dx}\right)-n^2(T-T_\infty)=0$

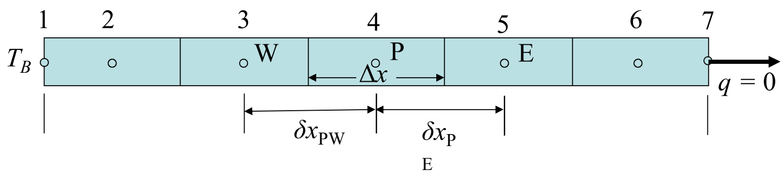
The general equation is: $\frac{d}{dx}(\Gamma\frac{d\phi}{dx})+S_\phi=0$ where $n^2=\frac{hP}{kA}$

$S\Delta V=S_u+S_p\phi_P$

$\Delta V=A\Delta x$

Comparing the above equations,

$\phi=T,\ \Gamma=1,\qquad S_u=n^2T_\infty\Delta V\qquad\qquad S_p=-n^2\Delta V$

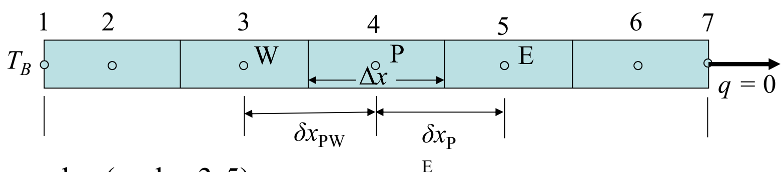


Solution is similar to the previous example. Find coefficients of

$$a_P\phi_P=a_W\phi_W+a_E\phi_E+S_u$$

(4-11)

General equation: $a_P\phi_P=a_W\phi_W+a_E\phi_E+S_u$



Interior nodes (nodes 3-5):

$S_u=n^2\Delta VT_\infty,\ S_p=-n^2\Delta V,$
$\Gamma=k$

a_W	a_E	a_P	S_p	S_u
$\frac{\Gamma_w A_w}{\delta x_{WP}}$	$\frac{\Gamma_e A_e}{\delta x_{PE}}$	$a_W+a_E-S_p$	$-n^2\Delta V$	$n^2\Delta VT_\infty$

For boundary node 2:

a_W	a_E	a_P	S_p	S_u
0	$\frac{\Gamma_e A_e}{\delta x_{PE}}$	$a_W+a_E-S_p$	$-n^2\Delta V-\frac{\Gamma_w A_w}{\Delta x/2}$	$n^2\Delta VT_\infty+\frac{\Gamma_w A_w}{\Delta x/2}T_B$

For boundary node 6:

a_W	a_E	a_P	S_p	S_u
$\frac{\Gamma_w A_w}{\delta x_{WP}}$	0	$a_W+a_E-S_p$	$-n^2\Delta V$	$n^2\Delta VT_\infty$

The solution is

$$\begin{bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \\ T_6 \\ T_7 \end{bmatrix} = \begin{bmatrix} 100 \\ 64.22 \\ 36.91 \\ 26.50 \\ 22.60 \\ 21.30 \\ 21.30 \end{bmatrix}$$

Comparison with the analytical solution

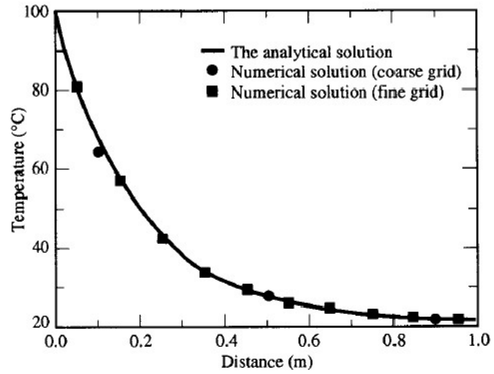
Node	*Distance*	*Finite volume solution*	*Analytical solution*	*Difference*	*Percentage Error*
2	0.1	64.22	68.52	4.30	6.27
3	0.3	36.91	37.86	0.95	2.51
4	0.5	26.50	26.61	0.11	0.41
5	0.7	22.60	22.53	−0.07	−0.31
6	0.9	21.30	21.21	−0.09	−0.42

Maximum error: 6.27%

The numerical solution can be improved by employing a finer grid.

Consider the same problem, but use 10 control volumes.

Comparison of the results is given as follows ig. 4.11 Comparison f numerical and nalytical results



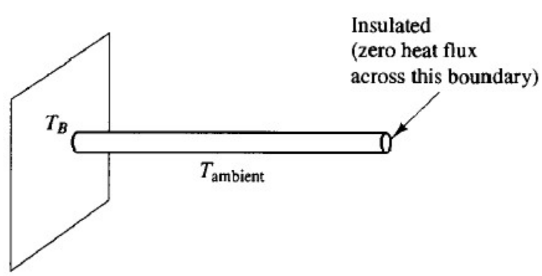
able 4.6

Node	*Distance*	*Finite volume solution*	*Analytical solution*	*Difference*	*Percentag error*
1	0.05	80.59	82.31	1.72	2.08
2	0.15	56.94	57.79	0.85	1.47
3	0.25	42.53	42.93	0.40	0.93
4	0.35	33.74	33.92	0.18	0.53
5	0.45	28.40	28.46	0.06	0.21
6	0.55	25.16	25.17	0.01	0.03
7	0.65	23.21	23.19	−0.02	−0.08
8	0.75	22.06	22.03	−0.03	−0.13
9	0.85	21.47	21.39	−0.08	−0.37
10	0.95	21.13	21.11	−0.02	−0.09

Maximum error: 2.08%

Homework 2:

Write a computer program to find the temperature distribution in the rod in example 4.3. Compare the results obtained using 10 and 50 points on a graph. Use the algorithm given in the pseudo program appearing in the following slide.

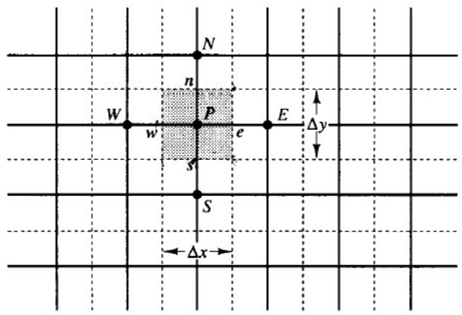


The geometry of example 4.3.

Finite Volume Method for Two-dimensional Diffusion Problems

Consider the two-dimensional steady state diffusion equation

$$\frac{\partial}{\partial x}\left(\Gamma\frac{\partial \phi}{\partial x}\right)+\frac{\partial}{\partial y}\left(\Gamma\frac{\partial \phi}{\partial y}\right)+S=0$$



Integrating the above equation over the CV,

$$\int_{\Delta V}\frac{\partial}{\partial x}\left(\Gamma\frac{\partial \phi}{\partial x}\right)dx\cdot dy+\int_{\Delta V}\frac{\partial}{\partial y}\left(\Gamma\frac{\partial \phi}{\partial y}dx\cdot dy\right)+\int_{\Delta V}S_\phi dV=0$$

Noting that $A_e = A_w = \Delta y$ and $A_n = A_s = \Delta x$, we obtain:

$$\left[\Gamma_e A_e\left(\frac{\partial \phi}{\partial x}\right)_e-\Gamma_w A_w\left(\frac{\partial \phi}{\partial x}\right)_w\right]+\left[\Gamma_n A_n\left(\frac{\partial \phi}{\partial y}\right)_n-\Gamma_s A_s\left(\frac{\partial \phi}{\partial y}\right)_s\right]+\bar{S}\Delta V=0 \qquad (4.53)$$

Equation (4.53) represents a balance of the generation of ϕ in a CV and the fluxes through its cell faces

Flux across the west face $=\Gamma_w A_w \left.\frac{\partial \phi}{\partial x}\right|_w = \Gamma_w A_w \frac{\phi_P-\phi_W}{\delta x_{WP}}$

Flux across the east face $=\Gamma_e A_e \left.\frac{\partial \phi}{\partial x}\right|_e = \Gamma_e A_e \frac{\phi_E-\phi_P}{\delta x_{PE}}$

Flux across the south face $=\Gamma_s A_s \left.\frac{\partial \phi}{\partial y}\right|_s = \Gamma_s A_s \frac{\phi_P-\phi_S}{\delta y_{SP}}$

Flux across the north face $=\Gamma_n A_n \left.\frac{\partial \phi}{\partial y}\right|_n = \Gamma_n A_n \frac{\phi_N-\phi_P}{\delta y_{PN}}$

By substitution of the above expressions into eqn. (4.53) we obtain

$$\Gamma_e A_e \frac{\phi_E - \phi_P}{\delta x_{PE}} - \Gamma_w A_w \frac{\phi_P - \phi_W}{\delta x_{WP}} + \Gamma_n A_n \frac{\phi_N - \phi_P}{\delta y_{PN}} - \Gamma_s A_s \frac{\phi_P - \phi_S}{\delta y_{SP}} + \bar{S}\Delta V = 0$$

Substituting the linearised form of the source term $\bar{S}\Delta V = S_u + S_P\phi_P$

$$\left(\frac{\Gamma_w A_w}{\delta x_{WP}} + \frac{\Gamma_e A_e}{\delta x_{PE}} + \frac{\Gamma_s A_s}{\delta y_{SP}} + \frac{\Gamma_n A_n}{\delta y_{PN}} - S_p\right)\phi_P$$
$$= \left(\frac{\Gamma_w A_w}{\delta x_{WP}}\right)\phi_W + \left(\frac{\Gamma_e A_e}{\delta x_{PE}}\right)\phi_E + \left(\frac{\Gamma_s A_s}{\delta y_{SP}}\right)\phi_S + \left(\frac{\Gamma_n A_n}{\delta y_{PN}}\right)\phi_N + S_u$$

This eqn can be written in the form:

$$a_P\phi_P = a_W\phi_W + a_E\phi_E + a_S\phi_S + a_N\phi_N + S_u$$

where

$A_w = A_e = \Delta y$
$A_s = A_n = \Delta x$

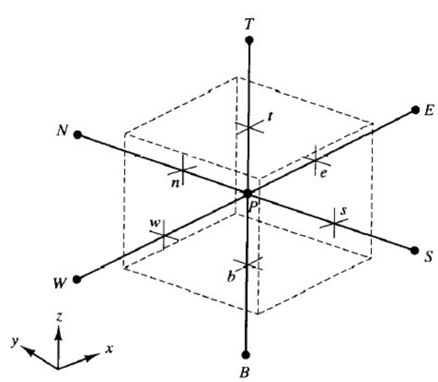
a_W	a_E	a_S	a_N	a_P
$\frac{\Gamma_w A_w}{\delta x_{WP}}$	$\frac{\Gamma_e A_e}{\delta x_{PE}}$	$\frac{\Gamma_s A_s}{\delta y_{SP}}$	$\frac{\Gamma_n A_n}{\delta y_{PN}}$	$a_W + a_E + a_S + a_N - S_p$

Finite Volume Method for Three-dimensional Diffusion Problems

Steady state diffusion in a 3D situation is governed by

$$\frac{\partial}{\partial x}\left(\Gamma\frac{\partial \phi}{\partial x}\right) + \frac{\partial}{\partial y}\left(\Gamma\frac{\partial \phi}{\partial y}\right) + \frac{\partial}{\partial z}\left(\Gamma\frac{\partial \phi}{\partial z}\right) + S = 0 \qquad (4.58)$$

A typical control volume is shown below.



Integration of eqn (4.58) over the control volume gives

$$\left[\Gamma_e A_e\left(\frac{\partial\phi}{\partial x}\right)_e - \Gamma_w A_w\left(\frac{\partial\phi}{\partial x}\right)_w\right] + \left[\Gamma_n A_n\left(\frac{\partial\phi}{\partial y}\right)_n - \Gamma_s A_s\left(\frac{\partial\phi}{\partial x}\right)_s\right]$$
$$+\left[\Gamma_t A_t\left(\frac{\partial\phi}{\partial z}\right)_t - \Gamma_b A_b\left(\frac{\partial\phi}{\partial z}\right)_b\right] + \bar{S}\Delta V = 0$$

which can be discretized as

$$\Gamma_e A_e \frac{\phi_E-\phi_P}{\delta x_{PE}} - \Gamma_w A_w \frac{\phi_P-\phi_W}{\delta x_{WP}} + \Gamma_n A_n \frac{\phi_N-\phi_P}{\delta y_{PN}} - \Gamma_s A_s \frac{\phi_P-\phi_S}{\delta y_{SP}}$$
$$+\Gamma_t A_t \frac{\phi_T-\phi_P}{\delta z_{PT}} - \Gamma_b A_b \frac{\phi_P-\phi_B}{\delta z_{BP}} + (S_u + S_p)\phi_P = 0$$

Rearranging $a_P\phi_P = a_W\phi_W + a_E\phi_E + a_S\phi_S + a_N\phi_N + a_B\phi_B + a_T\phi_T + S_u$

a_W	a_E	a_S	a_N	a_B	a_T	a_P
$\frac{\Gamma_w A_w}{\delta x_{WP}}$	$\frac{\Gamma_e A_e}{\delta x_{PE}}$	$\frac{\Gamma_s A_s}{\delta y_{SP}}$	$\frac{\Gamma_n A_n}{\delta y_{PN}}$	$\frac{\Gamma_b A_b}{\delta z_{BP}}$	$\frac{\Gamma_t A_t}{\delta y_{PT}}$	$a_W + a_E + a_S + a_N$ $+a_B + a_T - S_p$

Summary of Discretized Equations for Diffusion Problems

$$a_P\phi_P = \sum a_{nb}\phi_{nb} + S_u$$
$$a_P = \sum a_{nb} - S_p$$
source terms: $\bar{S}\Delta V = S_u + S_p\phi_P$

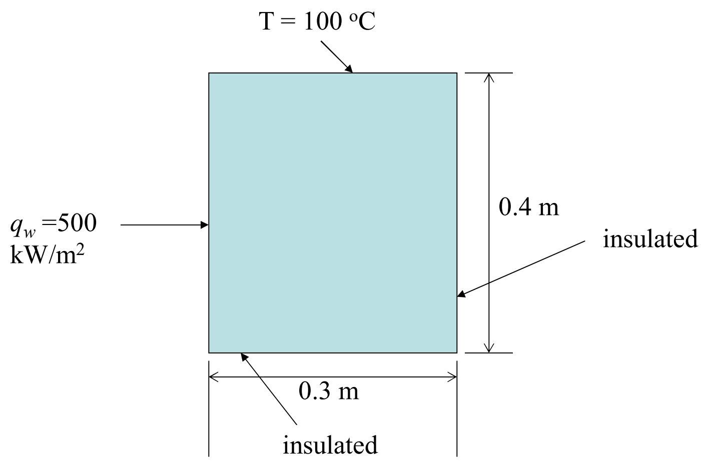
	a_W	a_E	a_S	a_N	a_B	a_T	a_P
1D	$\frac{\Gamma_w A_w}{\delta x_{WP}}$	$\frac{\Gamma_e A_e}{\delta x_{PE}}$					$a_W + a_E - S_p$
2D	$\frac{\Gamma_w A_w}{\delta x_{WP}}$	$\frac{\Gamma_e A_e}{\delta x_{PE}}$	$\frac{\Gamma_s A_s}{\delta y_{SP}}$	$\frac{\Gamma_n A_n}{\delta y_{PN}}$			$a_W + a_E + a_S + a_N$ $-S_p$
3D	$\frac{\Gamma_w A_w}{\delta x_{WP}}$	$\frac{\Gamma_e A_e}{\delta x_{PE}}$	$\frac{\Gamma_s A_s}{\delta y_{SP}}$	$\frac{\Gamma_n A_n}{\delta y_{PN}}$	$\frac{\Gamma_b A_b}{\delta z_{BP}}$	$\frac{\Gamma_t A_t}{\delta y_{PT}}$	$a_W + a_E + a_S + a_N$ $+a_B + a_T - S_p$

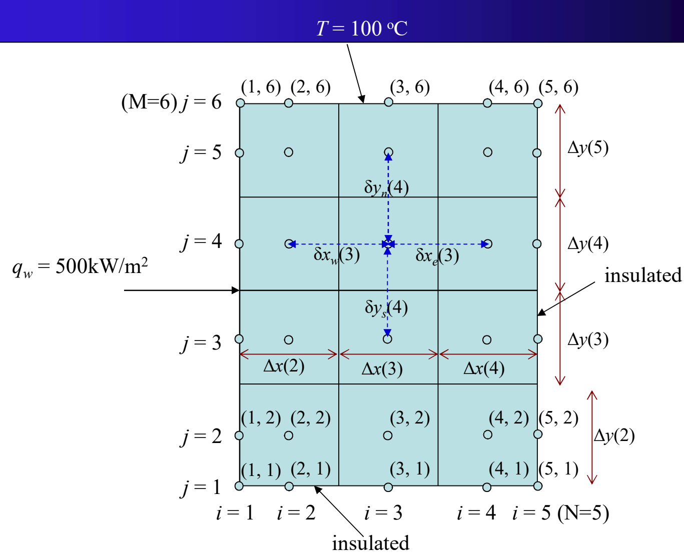
Example:

Consider a 2D plate

Thickness = 1cm, k = 1000W/m/K

Calculate the temperature distribution





First draw control volumes, with equal spacings

Then, place nodes at the center of the control volumes.

$\Delta x = L_x/N - 2 = 0.3/(5-2) = 0.1$, $\Delta y = L_y/M - 2 = 0.4/(6-2) = 0.1$

The governing equation is

$$\frac{\partial}{\partial x}\left(\Gamma\frac{\partial T}{\partial x}\right)+\frac{\partial}{\partial y}\left(\Gamma\frac{\partial T}{\partial y}\right)=0 \qquad (\Gamma = k)$$

which can be discretised as

$$a_P T_P = a_W T_W + a_E T_E + a_S T_S + a_N T_N + S_u$$

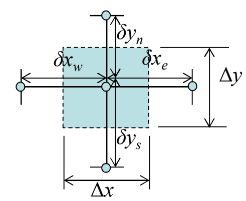
This equation is written for each node (i, j) in the domain

$$a_P(i,j)T(i,j) = a_W(i,j)T(i-1,j) + a_E(i,j)T(i+1,j) + a_S(i,j)T(i,j-1) + a_N(i,j)T(i,j+1) + S_u(i,j)$$

For interior points: $(i = 2-4, j = 2-6)$

$$a_W = \frac{\Gamma A_w}{\delta x_w}; \quad a_E = \frac{\Gamma A_e}{\delta x_e}; \quad a_S = \frac{\Gamma A_s}{\delta x_s}; \quad a_N = \frac{\Gamma A_n}{\delta x_n}$$

$$a_P = a_W + a_E + a_S + a_N - S_p \qquad S_p = 0, \quad S_u = 0$$



After finding a_P, a_E, a_W, a_N, a_S coefficients solve for all ϕ_P values using the Gauss-Seidel iteration method.

Repeat iterations until **scaled** residual norm becomes $R \leq \varepsilon$ where ε = tolerance (use ε = 1.E–6)

$$R = \frac{r}{\sum_j \sum_i |a_P(i,j) T_P(i,j)|}$$

where r is the residual norm defined as

$$r = \sum_j \sum_i |r(i,j)|$$

(Note the absolute value sign)

and

$$\begin{aligned} r(i,j) = a_W(i,j)T(i-1,j) + a_E(i,j)T(i+1,j) \\ + a_S(i,j)T(i,j-1) + a_N(i,j)T(i,j+1) \\ + S_u(i,j) - a_P(i,j)T(i,j) \end{aligned}$$

Fast Iterative Solvers for Linear Systems of Equations

Apart from TDMA, there are other iterative methods for solving the system of equations which are faster. Unlike TDMA, which solves the problem line by line, these iterative methods solves all equations simultaneously. As a result these methods are faster than TDMA. Some of the fast iterative methods are

1) **SIP** (strongly implicit procedure)
2) **MSIP** (modified SIP)
3) **CG** (Conjugate gradient method)
4) **BiCGSTAB** (bi-conjugate gradient stabilized method)

CG method is used for solving linear systems of equations which have a **symmetric** coefficient matrix. All other methods mentioned above are used for systems of equations involving **non-symmetric** coefficient matrices.

Homework 3:

Write a computer program to find the temperature distribution in the 2-D plate problem given in the previous example. Use the algorithm given in the pseudo program on the next page. (a) Use 5x6 grids (b) 51x51 grids and plot the temperature countours.

Thickness = 1cm, k = 1000W/m/K