Chapter 6: Solution Algorithms for Pressure-Velocity Coupling in Steady Flows

Ibrahim Sezai
Department of Mechanical Engineering
Eastern Mediterranean University

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Introduction

- The convection of a scalar variable ϕ depends on the magnitude and direction of the local velocity field.
- How to find flow field?
- Momentum equations can be derived from the general transport equation (2.39)

$$\frac{\partial(\rho\phi)}{\partial t} + div(\rho\phi\mathbf{u}) = div(\mu\, grad\, \phi) + S_\phi \qquad (6.1)$$

by replacing the variable ϕ by u, v and w.

- Let us consider the equations governing a two-dimensional, steady flow:

Introduction

- X-momentum equation

$$\frac{\partial}{\partial x}(\rho uu)+\frac{\partial}{\partial y}(\rho vu)=\frac{\partial}{\partial x}\left(\mu\frac{\partial u}{\partial x}\right)+\frac{\partial}{\partial y}\left(\mu\frac{\partial u}{\partial y}\right)+s_u \qquad (6.2)$$

- Y-momentum equation

$$\frac{\partial}{\partial x}(\rho uv)+\frac{\partial}{\partial y}(\rho vv)=\frac{\partial}{\partial x}\left(\mu\frac{\partial v}{\partial x}\right)+\frac{\partial}{\partial y}\left(\mu\frac{\partial v}{\partial y}\right)+s_v \qquad (6.3)$$

- Continuity equation

$$\frac{\partial}{\partial x}(\rho u)+\frac{\partial}{\partial y}(\rho v)=0 \qquad (6.4)$$

$s_u = -\partial p/\partial x$ for x-momentum equation

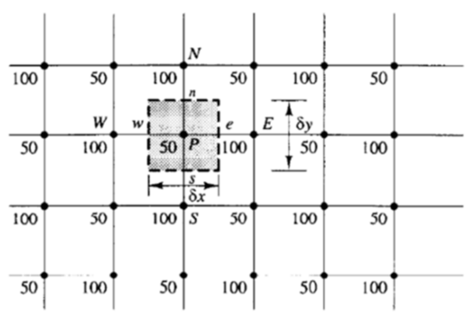
$s_v = -\partial p/\partial y$ for y-momentum equation

- The convective terms contain non-linear quantities.
- All three equations are intricately coupled.
- There is no equation for pressure.

The staggered grid

- Where to store the velocities?
- If the velocities and the pressures are both defined at the nodes of an ordinary CV a highly non-uniform pressure field can act like a uniform field in the discretized momentum equations.

A checker-board pressure field

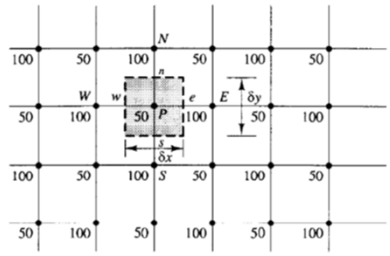


- Suppose that the pressure field is oscillatory as shown above

$$\frac{\partial p}{\partial x} = \frac{p_e - p_w}{\delta x} = \frac{\left(\frac{p_E + p_P}{2}\right) - \left(\frac{p_P + p_W}{2}\right)}{\delta x}$$

$$= \frac{p_E - p_W}{2\delta x}$$

$$\frac{\partial p}{\partial y} = \frac{p_N - p_S}{2\delta y}$$



- The pressure at the central node (P) does not appear in above equations. This gives zero pressure gradients at all nodal points indicating uniform pressure field. Not realistic.
- Solution: Use a staggered grid system for the velocity components.
- That is: evaluate scalar variables (p, ρ, T) at ordinary nodal points.
- But calculate velocity components (u, v) at cell faces which are staggered relative to nodal points.

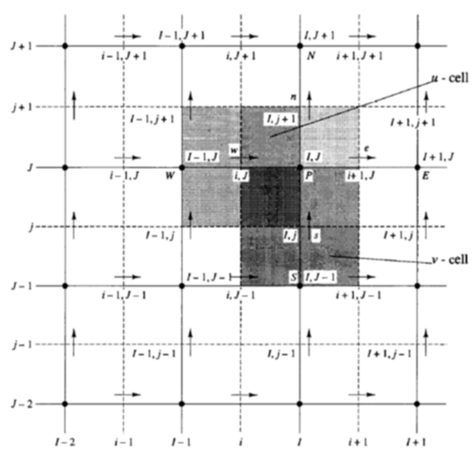
- CV's for u and v are different from the scalar CV's of p and T
- $u(i, j)$ is defined at west face of $p(i, j)$.
- $v(i, j)$ is defined at south face of $p(i, j)$.
- This is backward staggered system.
- For u-control volume:

$$\frac{\partial p}{\partial x} = \frac{p_P - p_W}{\delta x_u}$$

- For v-control volume:

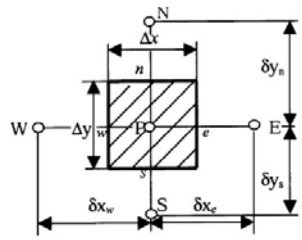
$$\frac{\partial p}{\partial y} = \frac{p_P - p_S}{\delta y_v}$$

- This arrangement gives non-zero pressure gradient terms. → Gives realistic behavior for pressure field.



Non-staggered (Collocated) Grid System

- The non-staggered grid system is complicated for unstructured or body-fitted mesh systems.
- Also, the storage of u,v,w and Pressure to four different locations is inefficient.
- In non-staggered grid system all variables are stored at the same location (point P).



- The problem of checker-board pressure field is avoided by calculating the cell face velocities from interpolation using the momentum equations (momentum interpolation method).

The momentum equations

- The discretised momentum equations at location P for a point (i, j) is

$$a_P\,\phi_P = a_E\phi_E + a_W\phi_W + a_N\phi_N + a_S\phi_S + S \qquad (6.5)$$

$$S = s_C^{body}V_P + S^{trans} + S^{dc} + S^{pres},$$

$$S^{trans} = a_P^o\phi_P^o,\quad a_P^o = \frac{\rho_P^0 V_P}{\Delta t}\ \text{(transient terms)},\qquad s^{body} = s_C^{body} + s_P^{body}\phi_P\ \text{(body forces per unit volume in the differential equation)}$$

$$a_E = \frac{\Gamma_e\Delta y}{\Delta x_e} + \max[-F_e,\ 0],\quad a_W = \frac{\Gamma_w\Delta y}{\Delta x_w} + \max[F_w,\ 0],\quad a_N = \frac{\Gamma_n\Delta x}{\Delta y_n} + \max[-F_n,\ 0],\quad a_S = \frac{\Gamma_s\Delta x}{\Delta y_s} + \max[F_s,\ 0]$$

$$a_P = a_E + a_W + a_N + a_S + \frac{\rho_P V_P}{\Delta t} - s_P^{body}\Delta x\Delta y + \Delta F,$$

$$\Delta F = F_e - F_w + F_n - F_s,$$

$$S^{pres} = \begin{cases} = -\dfrac{\partial p}{\partial x}\Delta V = -[(p_e - p_w)/\Delta x]\Delta V = -(p_e - p_w)\Delta y & \text{for x-momentum equation} \\ = -\dfrac{\partial p}{\partial y}\Delta V = -[(p_n - p_s)/\Delta y]\Delta V = -(p_n - p_s)\Delta x & \text{for y-momentum equation} \end{cases}$$

$$\begin{aligned} S^{dc} = &-\max[+F_e,\ 0](\phi_e - \phi_P) + \max[-F_e,\ 0](\phi_e - \phi_E) \\ &-\max[-F_w,\ 0](\phi_w - \phi_P) + \max[+F_w,\ 0](\phi_w - \phi_W) \\ &-\max[+F_n,\ 0](\phi_n - \phi_P) + \max[-F_n,\ 0](\phi_n - \phi_N) \\ &-\max[-F_s,\ 0](\phi_s - \phi_P) + \max[+F_s,\ 0](\phi_s - \phi_S) \end{aligned} \qquad (6.6)$$

$F_e = (\rho u)_e\Delta y$, $F_w = (\rho u)_w\Delta y$, $F_n = (\rho v)_n\Delta x$, $F_s = (\rho v)_s\Delta x$, (u_e, u_w, v_n, v_s are found by MIM method)

$\phi_e, \phi_w, \phi_n, \phi_s$ = face values found from a high order (higher than 1st order) convection scheme such as QUICK or CD

The momentum equations

- S momentum source term such as body forces
- The coefficients a_P, a_E, a_W a_N and a_S are calculated by upwind method.
- S^{dc} is the source term resulting from the adoption of the deferred correction method when any high order convection scheme, such as QUICK, is used in estimating the cell face value ϕ_f.
- The coefficients a_E, a_W etc. contain:
- 1) Convective flux per unit mass, F
- 2) Diffusive conductance, D

at control volume faces.

- It can be observed that the coefficients of the discretized x- and y-momentum equations are the same in collocated grid system, provided that the diffusion coefficient, Γ, is the same in x and y-momentum equations.
- In order to slow down the changes of dependent variables in consecutive solutions, an under-relaxation factor is introduced into the discretized equation (6.5) as follows:

$$\phi^{new} = \alpha_\phi \phi + (1-\alpha_\phi)\phi^{n-1} \qquad (6.7)$$

- where α_ϕ = under-relaxation factor

  ϕ^{n-1} = value of ϕ from the previous iteration
- The under-relaxed form of the general equation is

$$\frac{a_P^\phi}{\alpha_\phi}\phi_P = \left(a_E\phi_E + a_W\phi_W + a_N\phi_N + a_S\phi_S + S\right) + \frac{\left(1-\alpha_\phi\right)}{\alpha_\phi} a_P^\phi \phi_P^{n-1} \qquad (6.8)$$

- Separating the pressure gradient term from the source term,

$S = b + S^{pres}$

where

b = source term excluding the pressure gradient term

$$S^{pres} = -\frac{\partial p}{\partial x}\Delta V = -[(p_e - p_w)/\Delta x]\Delta V = -(p_e - p_w)\Delta y \quad \text{for x-momentum equation}$$

$$S^{pres} = -\frac{\partial p}{\partial y}\Delta V = -[(p_n - p_s)/\Delta y]\Delta V = -(p_n - p_s)\Delta x \quad \text{for y-momentum equation}$$

- Equation (6.8) becomes

$$\phi_P = \frac{\alpha_\phi}{a_P^\phi}\left(a_E\phi_E + a_W\phi_W + a_N\phi_N + a_S\phi_S + b_P\right) + \left(1-\alpha_\phi\right)\phi_P^{n-1} + \frac{\alpha_\phi S^{pres}}{a_P^\phi}$$

$$= \frac{\alpha_\phi}{a_P^\phi}\left(a_E\phi_E + a_W\phi_W + a_N\phi_N + a_S\phi_S + B_P\right) + \frac{\alpha_\phi S^{pres}}{a_P^\phi} \quad (6.9)$$

$$B_P = b_P + \frac{\left(1-\alpha_\phi\right)}{\alpha_\phi} a_P^\phi \phi_P^{n-1} \quad (6.10)$$

- Note that the term $\left(1-\alpha_\phi\right)\phi_P^{n-1}$ should be added to the source term in equations (6.5) and (6.6) if the equations are relaxed.

Momentum Interpolation Method (MIM)

- If ϕ stands for u in Eqn. (6.9), the velocity component at nodes P and E, can be written as

$$u_P = \frac{\alpha_u\left(\sum_i a_i u_i + B_p\right)_P}{\left(a_P^u\right)_P} - \frac{\alpha_u \Delta y\left(p_e - p_w\right)_P}{\left(a_P^u\right)_P} \quad (6.11)$$

$$u_E = \frac{\alpha_u\left(\sum_i a_i u_i + B_p\right)_E}{\left(a_P^u\right)_E} - \frac{\alpha_u \Delta y\left(p_e - p_w\right)_E}{\left(a_P^u\right)_E} \quad (6.12)$$

- and for the interface velocity at the cell face e

$$u_e = \frac{\alpha_u\left(\sum_i a_i u_i + B_p\right)_e}{\left(a_P^u\right)_e} - \frac{\alpha_u \Delta y\left(p_E - p_P\right)}{\left(a_P^u\right)_e} \quad (6.13)$$

- where the terms on the right-hand side with subscript e should be interpolated in an appropriate manner. The interface velocity at cell faces w, n, and s can be obtained similarly.
- In Rhie and Chow's momentum interpolation, the first term and $1/(a_p)_e$ in second term of the Eq. (13) are linearly interpolated from their counterparts in Eqs. (6.11) and (6.12):

Momentum interpolation method (MIM)

$$\left(\frac{\sum_i a_i u_i + B_p}{a_P^u}\right)_e = f_e^+\left(\frac{\sum_i a_i u_i + B_p}{a_P^u}\right)_E + \left(1-f_e^+\right)\left(\frac{\sum_i a_i u_i + B_p}{a_P^u}\right)_P \qquad (6.14)$$

$$\frac{1}{\left(a_P^u\right)_e} = f_e^+\frac{1}{\left(a_P^u\right)_E} + \left(1-f_e^+\right)\frac{1}{\left(a_P^u\right)_P} \qquad (6.15)$$

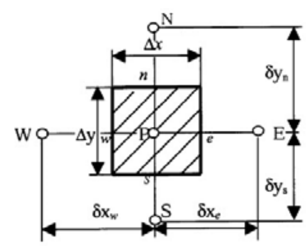
- where f_e^+ is a linear interpolation factor defined as $f_e^+ = \Delta x_P/(2\delta x_e)$
- Substituting $\left(\sum_i a_i u_i + B_p/a_P\right)$ terms from Eq's (6.11), (6.12) and (6.14) into Eq. (6.13) we obtain

$$u_e = \underbrace{\left[f_e^+ u_E + \left(1-f_e^+\right)u_P\right]}_{\text{linear interpolation term}} + \underbrace{\left\{\begin{array}{l} -\dfrac{\alpha_u \Delta y\left(p_E - p_P\right)}{\left(a_P^u\right)_e} + f_e^+\dfrac{\alpha_u \Delta y\left(p_e - p_w\right)_E}{\left(a_P^u\right)_E} \\ +\left(1-f_e^+\right)\dfrac{\alpha_u \Delta y\left(p_e - p_w\right)_P}{\left(a_P^u\right)_P}\end{array}\right\}}_{\text{correction term}} \qquad (6.16)$$

- The correction term has the function of smoothing the pressure field (remove the unrealistic pressure field).

- Equations (6.13) and (6.16) are essentially equivalent.
- Values of F and D for each of the faces e, w, n and s of the control volume at location (i, j):

$$F_e = (\rho u A)_e, \qquad F_w(i,j) = F_e(i-1,j)$$
$$u_e = \text{from MIM}, \qquad \rho_e = f_e^+\rho_E + (1-f_e^+)f_P$$
$$F_n = (\rho v A)_n, \qquad F_s(i,j) = F_n(i,j-1)$$
$$v_n = \text{from MIM}, \qquad \rho_n = f_n^+\rho_N + (1-f_n^+)f_P$$



$$\Gamma_e = f_e^+\Gamma_E + (1-f_e^+)\Gamma_P \qquad \Gamma_n = f_n^+\Gamma_N + (1-f_n^+)\Gamma_P$$

$$D_e = \frac{\Gamma_e A_e}{\delta x_e}, \quad D_n = \frac{\Gamma_e A_e}{\delta x_e}, \quad D_w(i,j) = D_e(i-1,j), \quad D_s(i,j) = D_n(i,j-1)$$

- If a property is unknown at a cell face then a suitable two-point average is used.
- u_e, v_n, … etc. in fluxes F_e, F_n, …at cell faces are calculated using MIM.
- The variables ϕ_e, ϕ_n, …etc. at cell faces in the deferred correction term S^{dc} (Eqn.(6.6)) are calculated using a convection scheme such as UPWIND or QUICK.
- During each iteration the u and v velocity component in F are those obtained from previous iteration.
- Hence, coefficients "a_e, a_n, …" are calculated using the u and v values from previous iteration.

- At each iteration level the values of F are computed using the u- and v-velocity components resulting from the previous iteration.
- Given a pressure field p, the momentum equations (6.2) and (6.3) can be written in the discretized form (6.5) for node P at each location (i, j) and then solved to obtain the velocity fields.
- If the pressure field is correct the resulting velocity field will satisfy continuity.
- As the pressure field is unknown, we need a method for calculating pressure.

The SIMPLE Algorithm

- SIMPLE (Semi-Implicit Method for Pressure-Linked Equations)
- For a guessed pressure field p^* the corresponding face velocity can be written using Eq. (6.13) as

$$u_e^* = \frac{\alpha_u \left(\sum_i a_i u_i^* + B_P^u\right)_e}{\left(a_P^u\right)_e} - \frac{\alpha_u \Delta y \left(p_E^* - p_P^*\right)}{\left(a_P^u\right)_e} + (1-\alpha_u) u_e^{n-1} \qquad (6.17)$$

- A similar equation can be written for the face velocity v_n^* .

$$v_n^* = \frac{\alpha_v \left(\sum_i a_i v_i^* + B_P^v\right)_n}{\left(a_P^v\right)_n} - \frac{\alpha_v \Delta x \left(p_N^* - p_P^*\right)}{\left(a_P^v\right)_n} + (1-\alpha_v) v_n^{n-1} \qquad (6.18)$$

- Let p', u', v' be the correction needed to correct the guessed pressure and velocity fields, i.e.

$$p = p^* + p' \qquad (6.19)$$

$$u_e = u_e^* + u_e' \qquad (6.20)$$

$$v_n = v_n^* + v_n' \qquad (6.21)$$

- Subtraction of eqn. (6.17) from (6.13) gives

$$u_e' = \frac{\alpha_u \left(\sum_i a_i u_i' + B_P^u\right)_e}{\left(a_P^u\right)_e} - \frac{\alpha_u \Delta y \left(p_E' - p_P'\right)}{\left(a_P^u\right)_e} \qquad (6.22)$$

- As an approximation, in SIMPLE method the first term in the above equation is neglected giving

$$u_e' = d_e^u \left(p_P' - p_E'\right) \qquad (6.23)$$

- where $d_e^u = \dfrac{\alpha_u A_e}{\left(a_P^u\right)_e}$, $\quad A_e = \Delta y$ (area of CV at face e)

- Similarly

$$v_n' = d_n^v\left(p_P' - p_N'\right) \qquad d_n^v = \frac{\alpha_v A_n}{\left(a_P^v\right)_n} \qquad (6.24)$$

- Then the corrected velocities become

$$u_e = u_e^* + d_e^u\left(p_P' - p_E'\right) \qquad (6.25)$$

$$v_n = v_n^* + d_n^v\left(p_P' - p_N'\right) \qquad (6.26)$$

- Discretizing the continuity equation (6.4) gives

$$(\rho u_e)\Delta y - (\rho u_w)\Delta y + (\rho v_n)\Delta x - (\rho v_s)\Delta x = 0 \qquad (6.27)$$

- Substituting the corrected face velocities such as that given by Eq's (6.24) and (6.25) into Eq. (6.27) gives

$$a_P p_P' = a_W p_W' + a_E p_E' + a_S p_S' + a_N p_N' + b \qquad (6.28)$$

$$a_P p_P' = a_W p_W' + a_E p_E' + a_S p_S' + a_N p_N' + b \qquad (6.28)$$

- where

$$a_E = (\rho A d)_e \quad a_W = (\rho A d)_w \quad a_N = (\rho A d)_n \quad a_S = (\rho A d)_s$$

$$a_P = a_w + a_e + a_s + a_n \qquad 6.29)$$

$$b = \left(\rho u^* A\right)_w - \left(\rho u^* A\right)_e + \left(\rho v^* A\right)_s - \left(\rho v^* A\right)_n$$

- Note that $(u^*)_w$, $(v^*)_s$,…, etc are calculated using MIM.
- After solving the p' field from Eq. (6.28) the face velocities are corrected using Eq.'s (6.25)and (6.26) and the pressure field is corrected by using

$$p = p^* + \alpha_p p' \qquad (6.30)$$

- α_p = pressure under-relaxation factor (chosen between 0 and 1).

- Similarly the nodal velocities are corrected using

$$u_P = u_P^* + d_P^u (p'_w - p'_e) \quad (6.31)$$

$$v_P = v_P^* + d_P^v (p'_s - p'_n) \quad (6.32)$$

- where

$$d_P^u = \frac{\alpha_u A_e}{(a_P^u)_P} \quad \text{and} \quad d_P^v = \frac{\alpha_v A_n}{(a_P^v)_P}$$

- The pressure corrections at the cell faces appearing in Eqs. (6.31) and (6.32) are calculated by linear interpolation from the nodal values as

$$p'_w = f_w^+ p'_W + (1 - f_w^+) p'_P \quad (6.33)$$

$$p'_e = f_e^+ p'_E + (1 - f_e^+) p'_P \quad (6.34)$$

$$p'_s = f_s^+ p'_S + (1 - f_s^+) p'_P \quad (6.35)$$

$$p'_n = f_n^+ p'_N + (1 - f_n^+) p'_P \quad (6.36)$$

Boundary Conditions for Pressure

- Since there is no equation for the pressure, no boundary conditions are needed for the pressure at the near boundary points.

- The pressure values at the boundaries can be calculated by linear extrapolation using the two near-boundary node pressures.

Boundary Conditions for Pressure Correction Equation

- When the velocities at the boundaries are known, there is no need to correct the velocities at the boundaries in the derivation of the pressure correction equation. For example if the velocity at the west boundary is known then for a control volume near the west boundary:

$$u_e = u_e^* + d_e^u\left(p'_P - p'_E\right) \qquad u_w = u_{wall}$$
$$v_n = v_n^* + d_n^v\left(p'_P - p'_N\right) \qquad v_s = v_s^* + d_s^v\left(p'_S - p'_P\right)$$

- Substituting above equations into the discretized continuity equation (6.27) we obtain the following pressure correction equation for a control volume near the west boundary

$$a_P p'_P = a_W p'_W + a_E p'_E + a_S p'_S + a_N p'_N + b \qquad (6.37)$$

- where $a_E = (\rho A d)_e \quad a_W = 0 \quad a_N = (\rho A d)_n \quad a_S = (\rho A d)_s$

$$b = \left(\rho u A\right)_{wall} - \left(\rho u^* A\right)_e + \left(\rho v^* A\right)_s - \left(\rho u^* A\right)_n \qquad (6.38)$$

- This formulation corresponds to Neuman b.c. ($\partial p'/\partial n = 0$) where n is normal to boundary.

- Comparing Eq.'s (6.37-6.38) with (6.28-6.29) for a near boundary control volume the same definition of the coefficients as used for the interior points can be used for a near boundary control volume by setting the corresponding coefficient (a_w in this case) to zero and using u_{wall} in the b term.
- As a result no value of pressure correction at the boundary (p'_w) is involved in this formulation.
- However, the value of the pressure correction is needed for correcting the nodal velocities near boundaries.
- For example, for correcting the u-velocity at a nodal point P near a west boundary, p'_w at the west boundary is needed in accordance with equation (6.31).
- This value can be obtained by using $\partial p'/\partial n = 0$ at the boundary, that is using $p'(1, j) = p'(2, j)$.

The SIMPLE Algorithm

- Step 1: Solve the discretized momentum equations
  $$\frac{a_P^u}{\alpha}u_P = \sum_i a_i u_i + b_p + (P_w - P_e)A_x + \frac{(1-\alpha)}{\alpha}a_P^u u^{n-1} \qquad \frac{a_P^v}{\alpha}v_P = \sum_i a_i v_i + b_p + (P_s - P_n)A_y + \frac{(1-\alpha)}{\alpha}a_P^v v^{n-1}$$
- Step 2: Calculate interface velocity u_e (Eqn's (6.16)) and similarly calculate v_n
  $$u_e = \left[f_e^+ u_E + (1-f_e^+)u_P\right] + \left\{-\frac{\alpha_u \Delta y(p_E - p_P)}{(a_P^u)_e} + f_e^+\frac{\alpha_u \Delta y(p_e - p_w)_E}{(a_P^u)_E} + (1-f_e^+)\frac{\alpha_u \Delta y(p_e - p_w)_P}{(a_P^u)_P}\right\}$$
- Use this velocity to find flux terms, F_e, F_w, etc…
- However, ϕ_e corresponding to u_e in S_{dc} term (Eqn. 6.6) is found from a convection scheme such as upwind or QUICK.
- Step3: Solve pressure correction equation (6.28)
  $$a_P^{p'} p'_P = a_W^{p'} p'_W + a_E^{p'} p'_E + a_S^{p'} p'_S + a_N^{p'} p'_N + b^{p'}$$
- Step 4: Correct pressure and velocities at points P using Eqn's (6.30), (6.31), (6.32)
  $$p = p^* + \alpha_p p' \qquad u_P = u_P^* + d_P^u(p'_w - p'_e) \qquad v_P = v_P^* + d_P^v(p'_s - p'_n)$$
- Step 5: Correct face velocities using equations (6.25) and (6.26):
  $$u_e = u_e^* + d_e^u(p'_P - p'_E) \qquad v_n = v_n^* + d_n^v(p'_P - p'_N)$$
- Step 6: Solve all other discretized transport equations (i.e. temperature)
  $$\frac{a_P^\phi}{\alpha}\phi_P = a_E^\phi\phi_E + a_W^\phi\phi_W + a_N^\phi\phi_N + a_S^\phi\phi_S + b_P^\phi + \frac{(1-\alpha)}{\alpha}a_P^\phi\phi^{n-1}$$
- Step 7: Repeat step 1 to 7 until convergence.