First Order Circuits

EEENG223

Circuit Theory I

First Order Circuits

A first-order circuit can only contain one energy storage element (a capacitor or an inductor). The circuit will also contain resistance. So there are two types of first-order circuits:

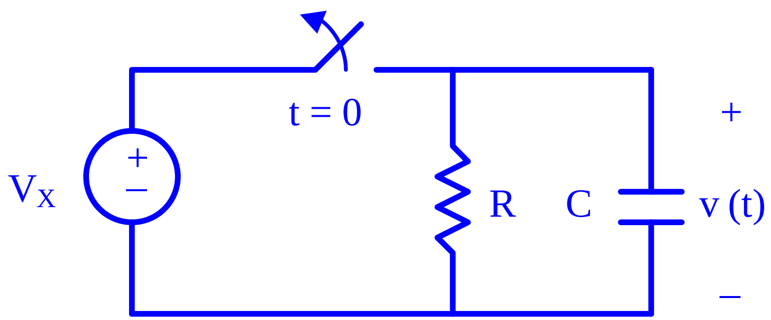
- RC circuit
- RL circuit

Source-Free Circuits

A ***source-free circuit*** is one where all independent sources have been disconnected from the circuit after some switch action. The voltages and currents in the circuit typically will have some transient response due to ***initial conditions*** (initial capacitor voltages and initial inductor currents). We will begin by analyzing source-free circuits as they are the simplest type. Later we will analyze circuits that also contain sources after the initial switch action.

SOURCE-FREE RC CIRCUITS

- Consider the RC circuit shown below. Note that it is source-free because no sources are connected to the circuit for t > 0. Use KCL to find the differential equation:



$$\frac{dv}{dt} + \frac{1}{RC} v(t) = 0 \quad \text{for } t \geq 0$$

- and solve the differential equation to show that:

$$v(t) = V_X e^{\frac{-t}{RC}} \quad \text{for } t \geq 0$$

SOURCE-FREE RC CIRCUITS

Checks on the solution

- Verify that the initial condition is satisfied.
- Show that the energy dissipated over all time by the resistor equals the initial energy stored in the capacitor.

First Order Circuits

General form of the D.E. and the response for a 1st-order source-free circuit

- In general, a first-order D.E. has the form:

$$\frac{dx}{dt} + \frac{1}{\tau}x(t) = 0 \quad \text{for } t \geq 0$$

Solving this differential equation (as we did with the RC circuit) yields:

$$x(t) = x(0)e^{\frac{-t}{\tau}} \quad \text{for } t \geq 0$$

where τ = (Greek letter "Tau") = ***time constant*** (in seconds)

Notes concerning τ :

1) for the previous RC circuit the DE was: $\frac{dv}{dt} + \frac{1}{RC}v(t) = 0 \quad \text{for } t \geq 0$

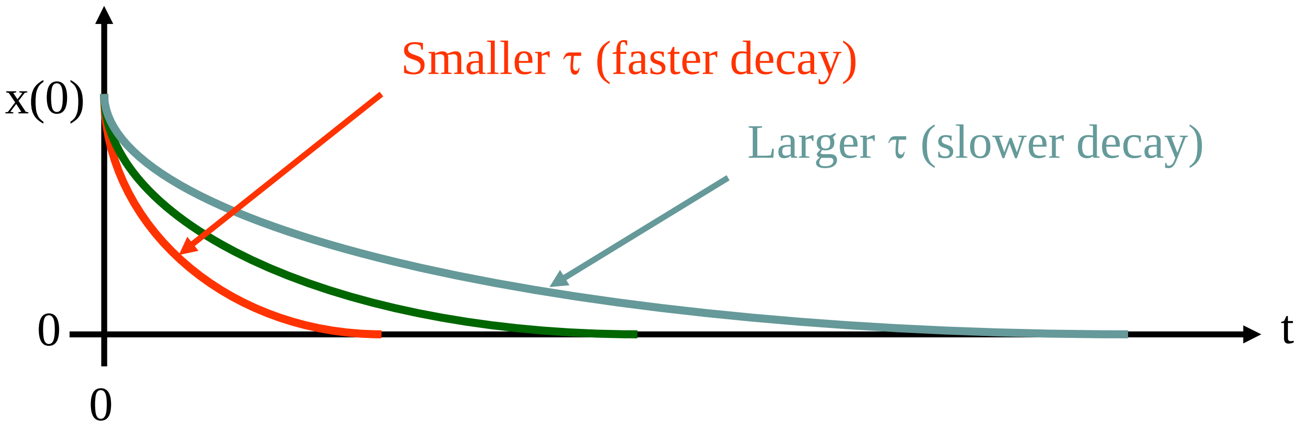
so

$$\tau = RC$$

(for an RC circuit)

2) τ is related to the rate of exponential decay in a circuit as shown below

$$x(t) = x(0)e^{\frac{-t}{\tau}}$$



3) It is typically easier to sketch a response in terms of multiples of τ than to be concerning with scaling of the graph (otherwise choosing an appropriate scale can be difficult). This is illustrated on the following page.

Example

The switch in the circuit shown had been closed for a long time and then opened at time t = 0.

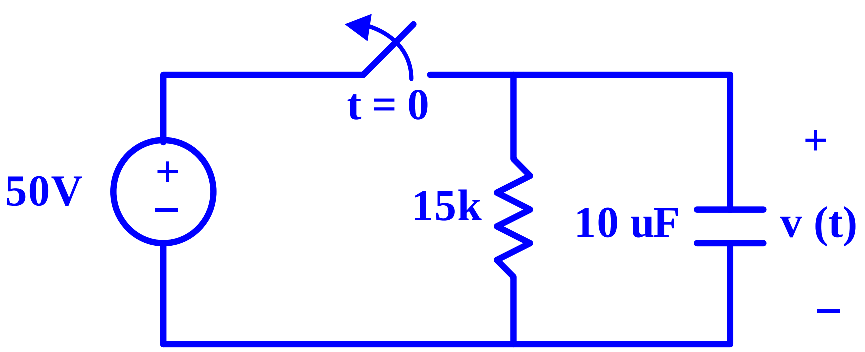
A) Determine an expression for v(t).

B) Graph v(t) versus t.

C) How long will it take for the capacitor to completely discharge?

D) Determine the capacitor voltage at time t = 100 ms.

E) Determine the time at which the capacitor voltage is 10V.

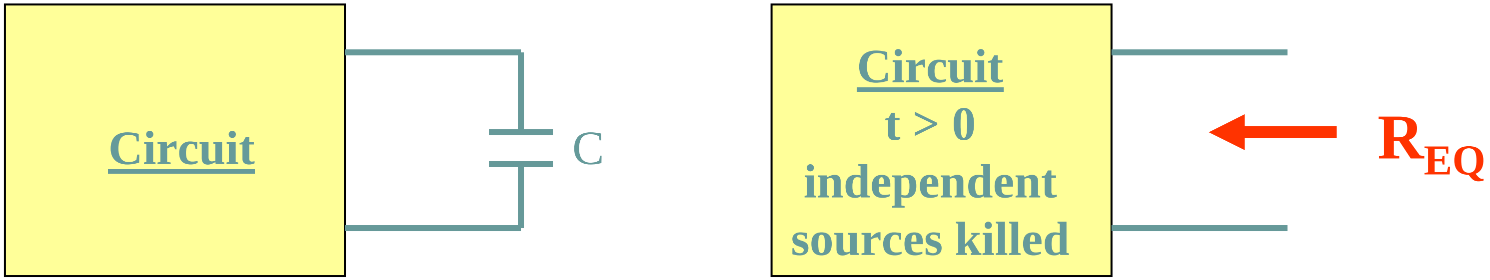


- **Equivalent Resistance seen by a Capacitor**
- For the RC circuit in the previous example, it was determined that $\tau = RC$. But what value of R should be used in circuits with multiple resistors?
- In general, a first-order RC circuit has the following time constant:

  - $\tau = R_{EQ}\, C$

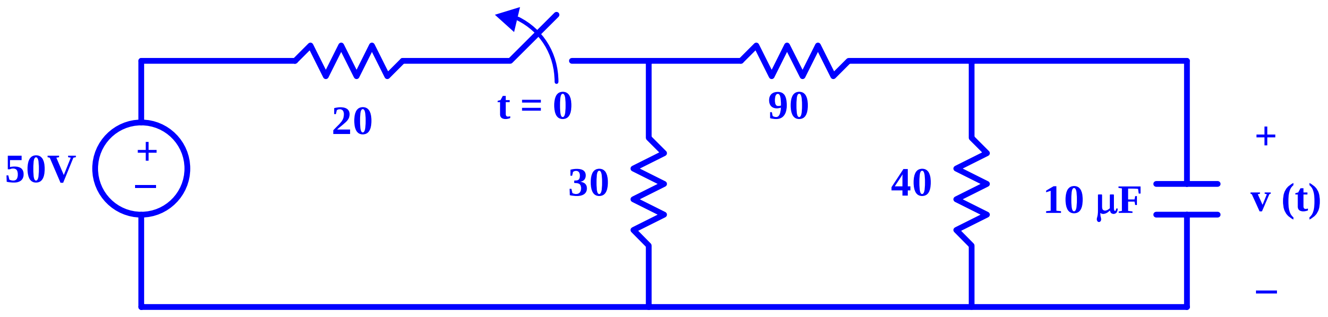
- where R_{eq} is the Thevenin resistance seen by the capacitor.
- More specifically,

$$R_{EQ} = R\Big|_{\substack{\text{seen from the terminals of the capacitor for } t > 0 \\ \text{with independent sources killed}}}$$



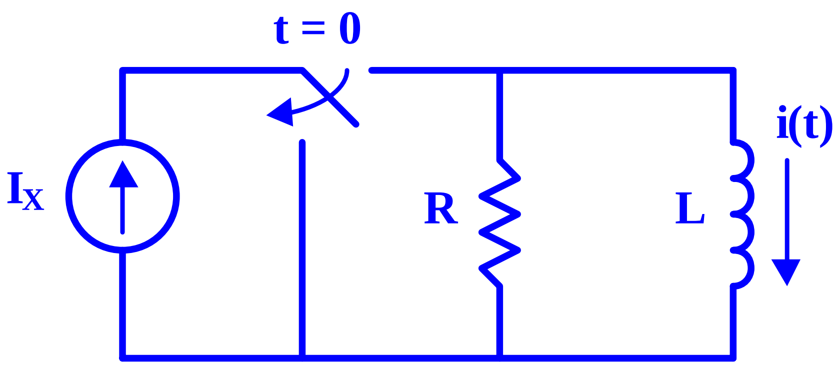
Example

Determine an expression for v(t). Graph v(t) versus t.



Source free RL Circuit

Consider the RL circuit shown below. Use KCL to find the differential equation:



$$\frac{di}{dt} + \frac{R}{L} i(t) = 0 \quad \text{for } t \geq 0$$

and use the general form of the solution to a first-order D.E. to show that:

$$\tau = L/R$$

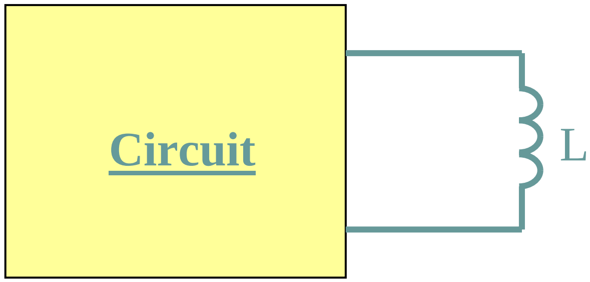
$$i(t) = I_X e^{\frac{-tR}{L}} \quad \text{for } t \geq 0$$

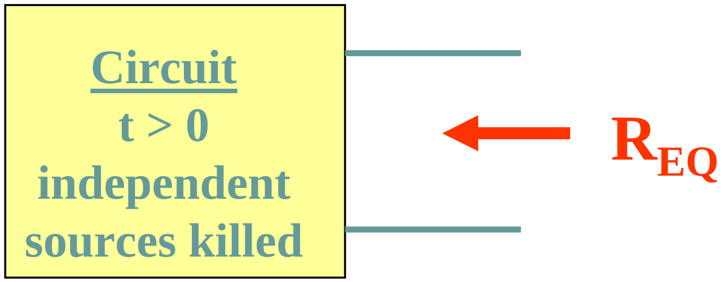
- **Equivalent Resistance seen by an Inductor**
- For the RL circuit in the previous example, it was determined that $\tau = L/R$. As with the RC circuit, the value of R should actually be the equivalent (or Thevenin) resistance seen by the inductor.
- In general, a first-order RL circuit has the following time constant:

$$\tau = \frac{L}{R_{EQ}}$$

where

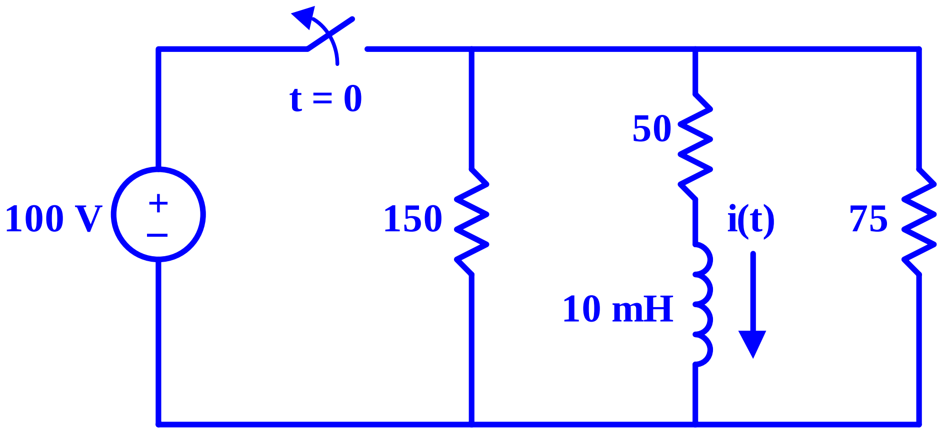
$$R_{EQ} = R\Big|_{\substack{\text{seen from the terminals of the inductor for } t > 0 \\ \text{with independent sources killed}}}$$





Example

Determine an expression for i(t). Sketch i(t) versus t.



First-order circuits with DC forcing functions:

In the last class we consider source-free circuits (circuits with no independent sources for t >0). Now we will consider circuits having DC forcing functions for t > 0 (i.e., circuits that do have independent DC sources for t > 0).

The general solution to a differential equation has two parts:

$x(t) = x_h + x_p$ = homogeneous solution + particular solution

or $x(t) = x_n + x_f$ = natural solution + forced solution

where x_h or x_n is due to the initial conditions in the circuit

and x_p or x_f is due to the forcing functions (independent voltage and current sources for t > 0).

x_p or x_f in general take on the “form” of the forcing functions, so

DC sources imply that the forced response function will be a constant (DC),

Sinusoidal sources imply that the forced response will be sinusoidal, etc.

Since we are only considering DC forcing functions in this chapter, we assume that

$x_f = B$ (a constant)

- Recall that a 1st-order source-free circuit had the form $Ae^{-t/\tau}$. Note that there was a natural response only since there were no forcing functions (sources) for t > 0. So the natural response was

  $x_n = Ae^{-t/\tau}$

- The complete response for 1st-order circuit with DC forcing functions therefore will have the form $x(t) = x_f + x_n$ or

$$x(t) = B + Ae^{-t/\tau}$$

- **The "Shortcut Method"**
- An easy way to find the constants B and A is to evaluate x(t) at 2 points. Two convenient points at $t = 0^-$ and $t = \infty$ since the circuit is under dc conditions at these two points. This approach is sometimes called the "shortcut method."
- So, $x(0) = B + Ae^0 = B + A$
- And $x(\infty) = B + Ae^{-\infty} = B$
- Show how this yields the following expression found in the text:

$$x(t) = x(\infty) + [x(0) - x(\infty)]e^{-t/\tau}$$

“Shortcut Method” - Procedure

The shortcut method will be the key method used in this chapter to analyze 1st-order circuit with DC forcing functions:

1) Analyze the circuit at $t = 0^-$: Find $x(0^-) = x(0^+)$, where $x = v_C$ or i_L .
2) Analyze the circuit at $t = \infty$: Find $x(\infty)$.
3) Find $\tau = R_{EQ}C$ or $\tau = L/R_{EQ}$.
4) Assume that $x(t)$ has the form $x(t) = B + Ae^{-t/\tau}$ and solve for B and A using $x(0)$ and $x(\infty)$.

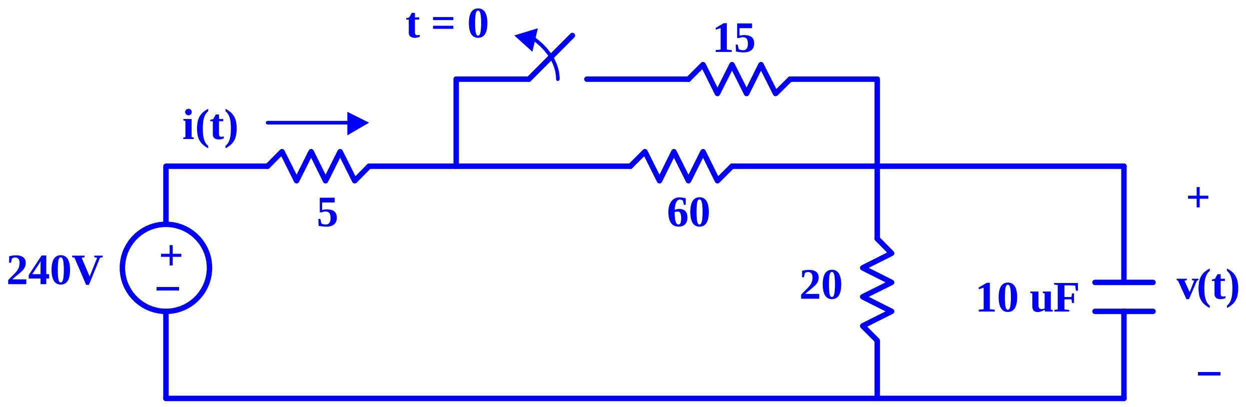
Notes:

The “shortcut method” also works for source-free circuits, but $x(\infty) = B = 0$ since the circuit is dead at $t = \infty$.

If variables other than v_C or i_L are needed, it is generally easiest to solve for v_C or i_L first and then use the result to find the desired variable.

Example

Find v(t) and i(t) for t $\geq$ 0.



Example

Find v(t) and i(t) for t $\geq$ 0.