EENG 428 Introduction to Robotics Laboratory
# Lab Session 1

Objective

In this Lab session, a review on linear algebra and matrix analysis is presented and described with the assist of the Matlab program to serve the purpose of this course.

1- Point and Vector Representation in Three-Dimensional Space

A point can be represented in space as a vector starting from the origin and ending at the point location using Cartesian coordinates as follow

$$P = a_x \hat{i} + a_y \hat{j} + a_z \hat{k}$$

Where $\hat{i}, \hat{j}$ and $\hat{k}$ are the unit vectors along x, y and z axes respectively. Besides, a_x, a_y and a_z are scalars represent the projections of the point on x, y and z respectively.
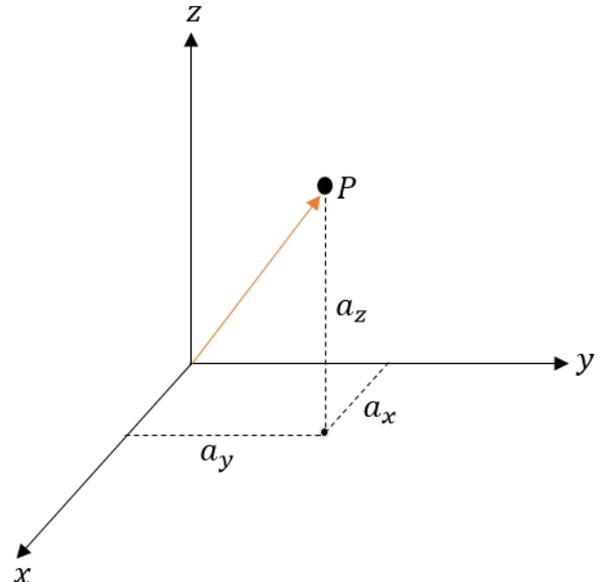
An alternative representation for a point in 3-D space using matrices is given by

$$P = [a_x \quad a_y \quad a_z]^T$$

The length (magnitude) of a vector P is defined as

$$\|P\|_2 = \sqrt{a_x{}^2 + a_y{}^2 + a_z{}^2}$$

Any non-zero vector can be normalized to have unity length to represent the direction by dividing each component by the magnitude of the vector.

Consider a vector drawn from point A to point B. Point A is called the initial point of the vector, and point B is called the terminal point. Symbolic notation for this vector is $\overrightarrow{AB}$

Two vectors are equivalent if they have the same magnitude and direction.
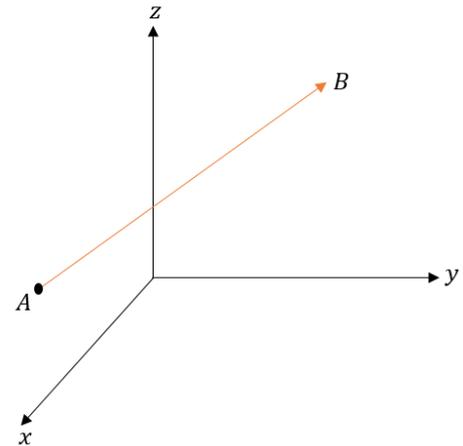
Example 1.1:

The Two Vector A and B Are Equivalent

```
A0=[2 4 0];  A1=[-1 3 0];
B0=[4 -1 1];  B1=[1 -2 1];
vectarrow(A0,A1); hold on;
vectarrow(B0,B1); hold on;
A=[(A1(1)-A0(1))  (A1(2)-A0(2))  (A1(3)-A0(3))]
B=[(B1(1)-B0(1))  (B1(2)-B0(2))  (B1(3)-B0(3))]
%magnitude of A
AM=sqrt((A(1)^2)+(A(2)^2)+(A(3)^2))
%magnitude of B
BM=sqrt((B(1)^2)+(B(2)^2)+(B(3)^2))
%unit vectors
VDirectionA=[A(1)/AM  A(2)/AM  A(3)/AM]
VDirectionB=[B(1)/BM  B(2)/BM B(3)/BM]
```
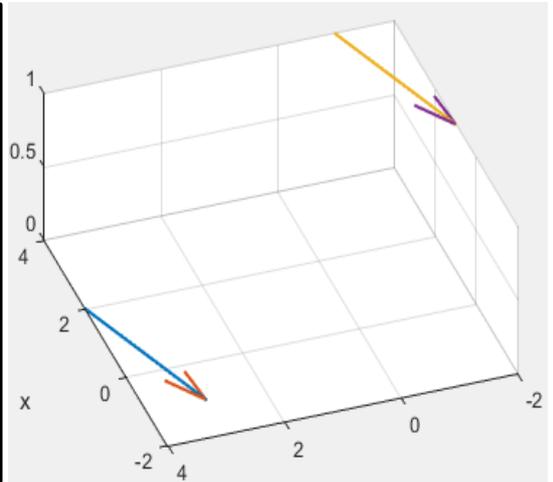

The representation of vectors can be slightly modified to include a scale factor w such that if x, y and z are divided by w we can reobtain the original vector. In other words, we can represent the vector $\boldsymbol{P} = [\boldsymbol{a_x} \quad \boldsymbol{a_y} \quad \boldsymbol{a_z}]^T$ in another form as $\boldsymbol{P} = [\boldsymbol{x} \quad \boldsymbol{y} \quad \boldsymbol{z} \quad \boldsymbol{w}]^T$. where $\boldsymbol{a_x} = \boldsymbol{x/w}$, $\boldsymbol{a_y} = \boldsymbol{y/w}$ and $\boldsymbol{a_z} = \boldsymbol{z/w}$ This representation will allow us to generalize a transformation matrix as we will discuss later.

Scale factor w can be any number. However, if $w = 0$, then a_x, a_y and a_z will be undefined. In this case, x, y, and z will represent a vector whose length is undefined, only the direction of vector remains definite. This means that a direction vector can be represented by a scale factor of $w = 0$, where the length is not important.

Example 1.2: Determine the unit directional vector along $P = 5\hat{\imath} + 2\hat{\jmath} + 1\hat{k}$

- The unit directional vector along P is $\hat{P} = \begin{bmatrix} \dfrac{\sqrt{30}}{6} & \dfrac{\sqrt{30}}{15} & \dfrac{\sqrt{30}}{30} \end{bmatrix}^T$

```
x=5;y=2;z=1;
xa=[0 x x x];
ya=[0 0 y y];
za=[0 0 0 z];
plot3(xa,ya,za,'--r','LineWidth',1.5);hold on;
plot3(x,y,z,'kO','LineWidth',2);hold on;grid on;
PM=sqrt(x^2+y^2+z^2) %magnitude of P
p0 = [0 0 0]; p1=[x/PM y/PM z/PM]
vectarrow(p0,p1); hold on;
XC = [x/PM x];
YC = [y/PM y];
ZC = [z/PM z];
plot3(XC,XY,XZ,'--k','LineWidth',0.5);hold on;
```
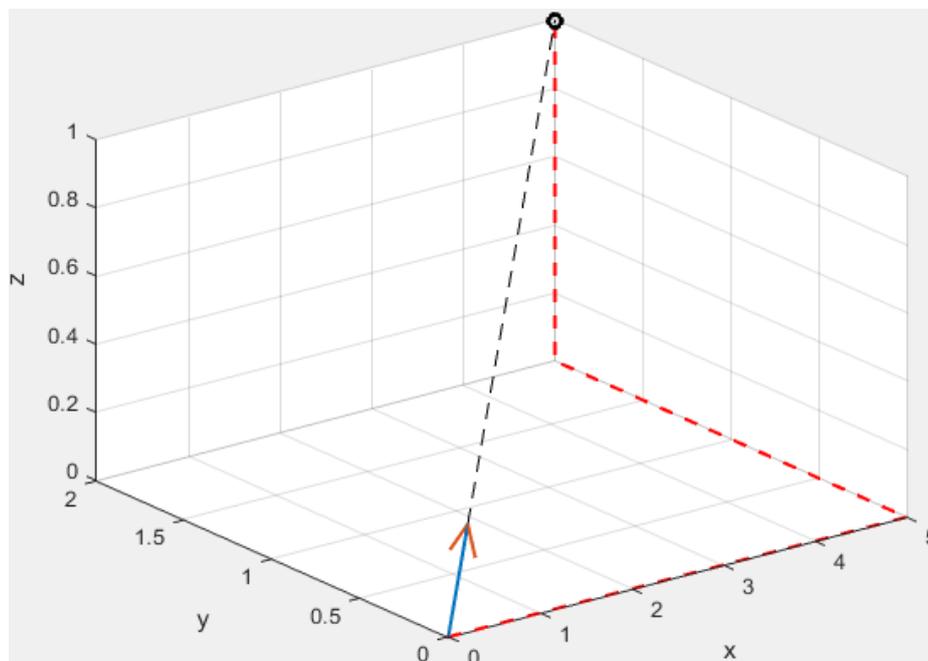
2-Vector-Vector Product Operation

2.1- Inner Product (Dot Product)

An operation that takes two vectors and returns a scalar quantity. the dot product between two vectors $A = a_x \hat{\imath} + a_y \hat{\jmath} + a_z \hat{k}$ and $B = b_x \hat{\imath} + b_y \hat{\jmath} + b_z \hat{k}$ can be defined as the sum of the products of the components of each vector as following

$$A.B = a_x * b_x + a_y * b_y + a_z * b_z$$

Geometrically, the dot product of two Euclidean Vectors **A** and **B** can be defined as the product of the magnitudes of the two vectors and the cosine of the angle between them. This statement clarified in the following equation
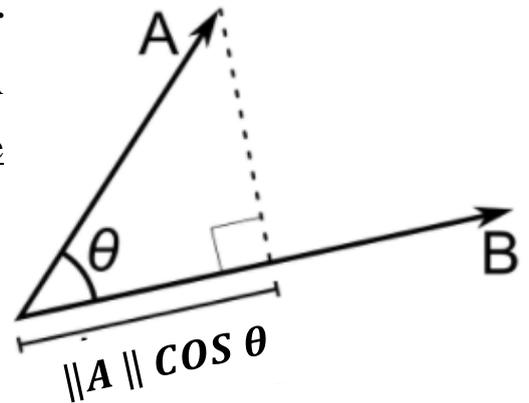
$$A.B = \|A\| \|B\| \cos \theta$$

Considering the two Vectors A and B. The **scalar projection** (or scalar component) of the Vector A on B is a scalar, equal to the <u>length of the orthogonal projection of A on B</u>.

$$A_B = \|A\| \cos \theta$$

- A_B is the scalar projection of A on B
- θ is the angle between A and B

Properties

- inner product is commutative in real vector subspaces.
- inner product between two non-zero vectors is zero if and only if the vectors are perpendicular (orthogonal).

Example 2.1: Consider the two vector $A = 1\hat{\imath} + 1\hat{\jmath} + 0\hat{k}$ and $B = 1.5\hat{\imath} + 0\hat{\jmath} + 0\hat{k}$

Finding the scalar component of the Vector A on B will be as the following

```
p0=[0 0 0];  A=[1 1 0];  B=[1.5
0 0];
vectarrow(p0,A); hold on;
vectarrow(p0,B); hold on;
AM=sqrt(A(1)^2+A(2)^2+A(3)^2);
%magnitude of A
BM=sqrt(B(1)^2+B(2)^2+B(3)^2);
%magnitude of B
ANGLE=dot(A,B)/(AM*BM); %Cosine
of the angle
DA=A/AM; %Direction of A
DB=B/BM;  %Direction of B
PAonB=AM*(ANGLE)*DB;
plot3([A(1) PAonB(1)],[A(2)
PAonB(2)],[A(3) PAonB(3)],'--
r','LineWidth',1);hold on;
plot3([0 PAonB(1)],[0
PAonB(2)],[0 PAonB(3)],'-
k','LineWidth',2)
```
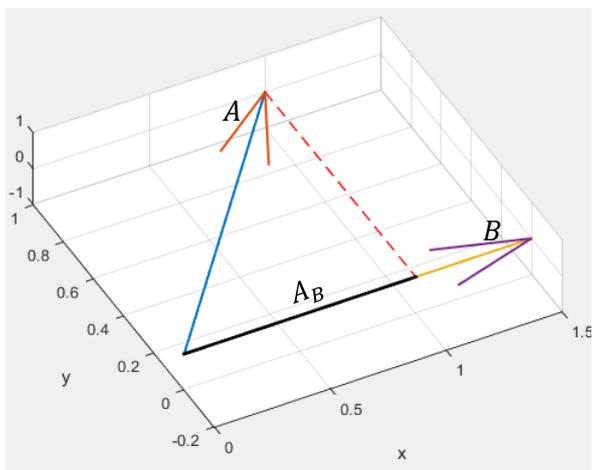

2.2- Outer Product (Cross Product)

The cross product of two vectors A and B is defined only in three-dimensional space as a vector C that is orthogonal to the plan that contain the vectors A and B.

 ➢ The direction of the Resultant vector C given by the right-hand rule

 ➢ The magnitude of the Resultant vector C equal to the area of the parallelogram that the vectors A and B span.
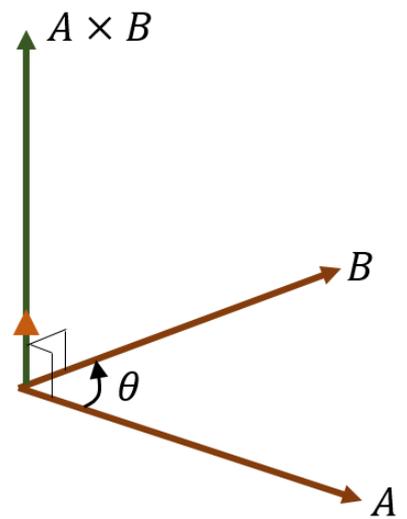
The cross product is defined by the formula

$$A \times B = \|A\|\|B\| \sin\theta\, \hat{n}$$

$\hat{n}$ is a unit vector orthogonal to the plane formed by A and B

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The cross product could be found algebraically for the vectors $A = a_x\,\hat{\imath} + a_y\,\hat{\jmath} + a_z\,\hat{k}$ and $B = b_x\,\hat{\imath} + b_y\,\hat{\jmath} + b_z\,\hat{k}$ as the normal matrix multiplication of one vector represented in a skew-symmetric matrix by the other

$$A \times B = \begin{bmatrix} 0 & -a_z & a_y \\ a_z & 0 & -a_x \\ -a_y & a_x & 0 \end{bmatrix} * \begin{bmatrix} b_x \\ b_y \\ b_z \end{bmatrix}$$

Properties

- The cross product is <u>not a commutative</u> operation $A \times B = -B \times A$

- inner product between two non-zero vectors is zero if and only if the vectors are parallel

Example 2.2: Consider the two vectors $A = 1.5\hat{\imath} + 0\hat{\jmath} + 0\hat{k}$ and $B = 1\hat{\imath} + 1\hat{\jmath} + 0\hat{k}$

Finding the scalar component of the Vector A on B will be as the following

```
p0=[0 0 0]; A=[1.5 0 0];
B=[1 1 0];
C1=cross(A,B);
vectarrow(p0,A); hold on;
vectarrow(p0,B); hold on;
vectarrow(p0,C1); hold on;

%C2=cross(B,A);
%vectarrow(p0,C2);hold on;
```
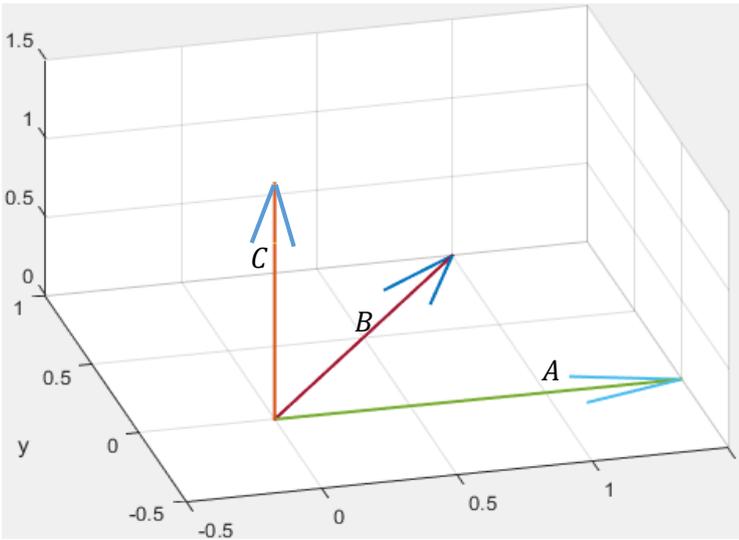

Exercise 1:

Consider the vectors $A = 1\hat{\imath} + 2\hat{\jmath} + 1\hat{k}$, $B = 2\hat{\imath} + 1\hat{\jmath} - 1\hat{k}$ and $C = 2\hat{\imath} - 2\hat{\jmath} - 4\hat{k}$

- Give a scaled representation for each of these vectors using the scalars
- Find a unit vector to represent the direction of a, b and c
- Find the result of $(A \times B).(A - C)$
- Verify your answers using matlab

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3- Moore-Penrose Pseudo Inverse

In mathematics, a system of linear equations (or linear system) is a collection of two or more linear equations involving set of variables. A general system of m linear equations with n unknowns can be written as

$$a_{11}\,x_1 + a_{12}\,x_2 + \cdots + a_{1n}\,x_n = b_1$$
$$a_{21}\,x_1 + a_{22}\,x_2 + \cdots + a_{2n}\,x_n = b_2$$
$$\vdots$$
$$a_{m1}\,x_1 + a_{m2}\,x_2 + \cdots + a_{mn}\,x_n = b_m$$

Where $x_1, x_2, \dots x_n$ are unknown, $a_{11}, a_{12}, \dots a_{mn}$ are the coefficients and $b_1, b_1, \dots b_1$ are the constant terms. The matrix representation form of the system given as

$$A\,x = b$$

Where A is a $m \times n$ matrix, x is a column vector with n entries, and b is a column vector with m entries

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \quad , \quad x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \quad \text{and} \quad b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

A solution of a linear system is an assignment of values to the variables $x_1, x_2, \dots x_n$ such that each of the equations is satisfied. A linear system may behave in any one of three possible ways: The system has **infinitely many solutions**, the system has **a single unique solution** and the system has **no solution**.

In mathematics, a pseudoinverse (Moore–Penrose inverse) A⁺ of a matrix A is a generalization of the inverse matrix solutions. It is proved that pseudoinverse A+ of the matrix A is unique matrix.

$$A^+ = \lim_{\delta \to 0}(A^T A + \delta^2 I)^{-1} A^T = \lim_{\delta \to 0} A^T (A\,A^T + \delta^2 I)^{-1}$$

Moore and Penrose showed that there is a general solution (Moore-Penrose solution) to the system of equations $A\,x = b$ of the form $x = A^+\,b$

Considering the matrix $A \in R_r^{m \times n}$

1- If $m > n$, in this case there are **more constraining equations than variables** (which corresponds to a **kinematically insufficient manipulator**). Hence, it is **not generally possible to find a solution** to these equations. The pseudoinverse gives the solution that **minimizes** the quantity $\|b - Ax\|$

Example 3.1

Determine the **General** solution (Moore-Penrose solution) to the system of equations illustrated below

$$x_1 + x_2 = 1$$
$$2x_1 + 3x_2 = 3$$
$$7x_1 + 4x_2 = 4$$

$$x_1 + x_2 = 1$$
$$2x_1 + 3x_2 = 3$$
$$7x_1 + 4x_2 = 2$$

```
clear all
A=[1 1 ;2 3; 7 4]
Aplus=pinv(A) %A+
x=Aplus*[1;3;4] %solution
E1=1*x(1)+ 1*x(2)
E2=2*x(1)+ 3*x(2)
E3=7*x(1)+ 4*x(2)
```

```
clear all
A=[1 1 ;2 3; 7 4]
Aplus=pinv(A) %A+
x=Aplus*[1;3;2] %solution
E1=1*x(1)+ 1*x(2)
E2=2*x(1)+ 3*x(2)
E3=7*x(1)+ 4*x(2)
```

>> Aplus =

  -0.0503 -0.2961   0.2346

   0.1061   0.5140 -0.1620

>> x =

   0.0   1.00

>> E1 = 1   E2 = 3   E3 = 4

>> Aplus =

  -0.0503 -0.2961   0.2346

   0.1061   0.5140 -0.1620

>> x =

  -0.4693   1.3240

>> E1 = 0.8547   E2 = 3.0335   E3 = 2.0112

2- If $m < n$ (which corresponds to a **kinematically redundant manipulator**). In this case, there **may be** **more than one possible solution** (generally an infinite number of solutions). The Moore-Penrose solution is the particular solution whose vector 2-norm $\|x\|$ is minimal

Example 3.2

Determine the **General** solution (Moore-Penrose solution) to the system of equations illustrated below

$$x_1 + 2x_2 + x3 = 2$$
$$2x_1 + 2x_2 + 3x3 = 3$$

```
clear all
A=[1 2 1 ;2 2 3]
Aplus=pinv(A)  %A+
x=Aplus*[2;3] % the solution
E1=1*x(1)+ 2*x(2)+ 1*x(3)
E2=2*x(1)+ 2*x(2)+ 3*x(3)
Vnorm=norm(x,2)
```

>> Aplus =

  -0.0476   0.1429

   0.7619 -0.2857

  -0.4762   0.4286

>> x =

0.3333 0.6667 0.3333

 >> E1 = 2   E2 = 3

Vnorm = **0.8165**

Even though $x = [1\ 0.5\ 0]$ whose vector 2-norm is 1.118 represent a solution to the given system, but the this solution is not **General solution** that is the particular solution whose 2-norm $\|x\|$ is minimal

3- If $m = n$ and the matrix A is invertible (**square, full rank matrix**) then the pseudoinverse identical to the normal inverse $A^+ = A^{-1}$. Even if the matrix A is not invertible (square but not full rank) in this case the determinant of the matrix A is zero subsequently there is no possible way to find the inversion of the matrix A in the normal way. The More-Penrose pseudoinverse can be used in such cases.

Properties

- If a matrix A is not-invertible, but $A^T A$ is invertible, the pseudo-inverse will be easy to find (respectively), and is called as the left inverse

$$A^+ = (A^T A)^{-1} A^T$$

- If a matrix A is not-invertible, but $A A^T$ is invertible, the pseudo-inverse will be easy to find (respectively), and is called as the right inverse.

$$A^+ = A^T (A A^T)^{-1}$$

Exercise 2:

Find by hand the pseudo-inverse of the Matrix $A = \begin{bmatrix} 1 & 1 \\ 3 & 3 \end{bmatrix}$

References:

1- Niku, S. (2010). *Introduction to robotics*. John Wiley & Sons.

2- Laub, A. J. (2005). *Matrix analysis for scientists and engineers*(Vol. 91). Siam.

3- Rotation Matrix (2019). In Wikipedia. Retrieved from

https://en.wikipedia.org/wiki/Rotation_matrix

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