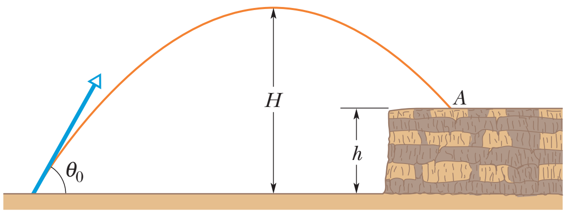
Tutorial 4: B: Motion in Two Dimensions

1. A motorist drives south at 20 m/s for 3 min, then turns west and travels at 25m/s for 2 min, and finally travels northwest at 30 m/s for 1 min. Let the positive x axis point east. For this 6 min trip, find
   (a) The total vector displacement,
   (b) The average speed, and
   (c) The average velocity.
2. A fish swimming in a horizontal plane has velocity $\vec{v}_i = (4\hat{\mathbf{i}} + 1\hat{\mathbf{j}})$m/s at a point in the ocean where the position relative to a certain rock is $\vec{r}_i = (10\hat{\mathbf{i}} - 4\hat{\mathbf{j}})$m. After the fish swims with constant acceleration for 20 s, its velocity is $\vec{v} = (20\hat{\mathbf{i}} - 5\hat{\mathbf{j}})$m/s
   (a) What are the components of the acceleration of the fish?
   (b) What is the direction of the acceleration with respect to the unit vector $\hat{\mathbf{i}}$?
   (c) If the fish maintains constant acceleration, where is it at $t = 25$s and in what direction is it moving?
3. An electron's position is given by $\vec{r} = 3t\hat{\mathbf{i}} - 4t^2\hat{\mathbf{j}} + 2\hat{\mathbf{k}}$, with t in seconds and $\vec{r}$ in meters.
   (a) In unit-vector notation, what is the electron's velocity $v\vec{(t)}$
   (b) What is $\vec{v}(t)$at $t = 2$s in unit-vector notation?
   (c) What is the magnitude of $\vec{v}$?
   (d) What is the angle of $\vec{v}$ relative to the positive direction of the x axis?
4. A plane flies 483 km east from city A to city B in 45 min and then 966 km south from city B to city C in 1.50 h. For the total trip, what are the
   (a) Magnitude and
   (b) Direction of the plane's displacement,
   (c) The magnitude and
   (d) Direction of its average velocity, and
   (e) Its average speed?

5. The position vector $\vec{r(t)}$ of a particle moving in the xy-plane is given by $\vec{r(t)} = (2t^3 - 5t)\hat{\mathbf{i}} + (6 - 7t^4)\hat{\mathbf{j}}$ with $\vec{r(t)}$ in meters and t in seconds.
   - (a) Calculate the position vector in $t = 0$ and $t = 1s$?
   - (b) Calculate the displacement of the particle between $t = 0$ and $t = 1s$?
   - (c) Calculate the average velocity between $t = 0$ and $t = 1s$?
   - (d) Calculate the instantaneous velocity and the acceleration of the particle at $t = 1s$?

6. A small ball rolls horizontally off the edge of a tabletop that is 1.20 m high. It strikes the floor at a point 1.52 m horizontally from the table edge.
   - (a) How long is the ball in the air?
   - (b) What is its speed at the instant it leaves the table?

7. In the figure below, a stone is projected at a cliff of height h with an initial speed of 42 m/s directed at angle $\theta_0 = 60°$ above the horizontal. The stone strikes at A, 5.50 s after launching.



Find

   - (a) The height h of the cliff,
   - (b) The speed of the stone just before impact at A, and
   - (c) The maximum height H reached above the ground.

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Solutions

1.

(a) First we have to calculate the position vectors, then we can calculate the net displacement:
In the first part of the trip the motorist travels with a constant velocity of 20 m/s in negative y direction, therefore the motorist's position vector after 3 min is:

$$\vec{r_1} = -20\,\hat{\mathbf{j}}\,\mathrm{m/s} \cdot 3\,\mathrm{min} = -20\,\hat{\mathbf{j}}\,\mathrm{m/s} \cdot 180\,\mathrm{s} = -3.60 \times 10^3\,\hat{\mathbf{j}}\,\mathrm{mm}$$

In the second part of the trip the motorist travels for 2.00 min with a constant speed of 25 m/s to the west, i.e. in negative x-direction, as the positive x direction points to the east. Therefore the displacement for the second part of the trip :

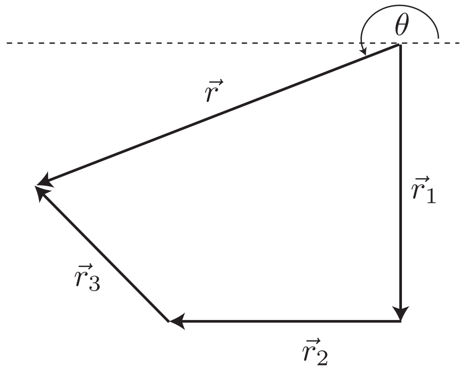
$$\vec{r_2} = -25\,\hat{\mathbf{i}}\,\frac{\mathrm{m}}{\mathrm{s}} 2\,\mathrm{min} = -25\,\hat{\mathbf{i}}\,\frac{\mathrm{m}}{\mathrm{s}} 120\,\mathrm{s} = -3 \times 10^3\,\hat{\mathbf{i}}\,\mathrm{m}$$

In the third part of the trip the motorist travels with a constant speed of $30\,\frac{\mathrm{m}}{\mathrm{s}}$ in northwest direction for 1 m. Therefore the displacement for the third part of the trip is:

$$\vec{r_3} = (30\,\hat{\mathbf{i}}\,\frac{\mathrm{m}}{\mathrm{s}}\cos(135°) + 30\,\hat{\mathbf{j}}\,\frac{\mathrm{m}}{\mathrm{s}}\sin(135°))1\,\mathrm{min} =$$
$$= (-21.21\,\hat{\mathbf{i}}\,\frac{\mathrm{m}}{\mathrm{s}} + 21.21\,\hat{\mathbf{j}}\,\frac{\mathrm{m}}{\mathrm{s}})60.0\,\mathrm{s} = (-1.27\,\hat{\mathbf{i}} \times 10^3\,\mathrm{m} + 1.27\,\hat{\mathbf{j}} \times 10^3\,\mathrm{m})$$

So finally the net displacement after 6 min is

$$\vec{r} = \vec{r_1} + \vec{r_2} + \vec{r_3} =$$
$$= (-3.60\,\hat{\mathbf{j}} \times 10^3\,\mathrm{m}) + (-3\,\hat{\mathbf{i}} \times 10^3\,\mathrm{m}) + (-1.27\,\hat{\mathbf{i}} \times 10^3\,\mathrm{m} + 1.27\,\hat{\mathbf{j}} \times 10^3\,\mathrm{m}) =$$
$$= (-4.27\,\hat{\mathbf{i}} \times 10^3\,\mathrm{m} - 2.33\,\hat{\mathbf{j}} \times 10^3\,\mathrm{m}) = (-4.27\,\hat{\mathbf{i}} - 2.33\,\hat{\mathbf{j}})\,\mathrm{km}$$



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The magnitude of the net displacement is then

$$r = \sqrt{r_x^2 + r_y^2} = \sqrt{(-4.27\,\text{km})^2 + (-2.33\,\text{km})^2} = 4.86\,\text{km}$$

The direction of the net displacement vector is then

$$\theta = \tan^{-1}\left(\frac{-2.33\,\text{km}}{-4.27\,\text{km}}\right) = 28.62^\circ$$

$$\theta_3 = 28.62^\circ + 180^\circ = 208.62^\circ$$

So the direction is 28.6° south of west or 208.62° with respect to the positive x-axis.

(b) In order to determine the average speed we have to sum up the magnitude of the displacements and finally divide the sum by the total amount of time that has passed. Therefore we get for the average speed, denoted by $v_{av.speed}$

$$v_{av.speed} = \frac{|\vec{r}_1| + |\vec{r}_1| + |\vec{r}_3|}{\Delta t} =$$
$$= \frac{3.60\,\text{km} + 3\,\text{km} + \sqrt{(-1.27\,\text{km})^2 + (1.27\,\text{km})^2}}{360\,\text{s}} = 0.0233\,\text{km/s} = 23.32\,\text{m/s}$$

(c) We get the magnitude of the average velocity as the ratio of the length of the displacement vector over the time that has passed.

$$v_{av} = \frac{|\vec{r}|}{\Delta t} = \frac{4.86 \times 10^3\,\text{m}}{360\,\text{s}} = 13.50\,\text{m/s}$$

The direction of the average velocity vector is in direction of the net displacement vector $\vec{r}$.

2.

(a)

$$\vec{v}(t) = \vec{v}_0 + \vec{a}t \quad \Rightarrow \vec{a} = \frac{1}{t}(\vec{v}(t) - \vec{v}_0)$$

$$\vec{a} = \frac{1}{20\,\text{s}}\left((20\hat{\mathbf{i}} - 5\hat{\mathbf{j}})\text{m/s} - (4\hat{\mathbf{i}} + 1\hat{\mathbf{j}})\text{m/s}\right) =$$
$$= \frac{1}{20\,\text{s}}((20-4)\hat{\mathbf{i}} + (-5-1)\hat{\mathbf{j}})\text{m/s} = \frac{1}{20\,\text{s}}(16\hat{\mathbf{i}} - 6\hat{\mathbf{j}})\text{m/s} =$$
$$= (0.80\hat{\mathbf{i}} - 0.30\hat{\mathbf{j}})\text{m/s}^2$$

So with matching technique we get for the components of the acceleration of the fish:

$$a_x = 0.80\,\text{m/s}^2, \qquad a_y = -0.30\,\text{m/s}^2.$$

(b) The direction is then

$$\theta = \tan^{-1}\left(\frac{-0.300\,\text{m/s}^2}{0.800\,\text{m/s}^2}\right) = -20.56^\circ$$

$$\theta_4 = -20.56^\circ + 360^\circ = 339.44^\circ$$

The acceleration vector has the direction 339.44° with respect to the positive x-axis

(c) In order to determine the position at $t = 25\,\text{s}$

$$\vec{r(t)} = \vec{r}_0 + \vec{v}_0 t + \frac{1}{2}\vec{a}t^2$$

$$\vec{r(t)} = 10\,\hat{\mathbf{i}}m - 4\,\hat{\mathbf{j}}m + 4\,\hat{\mathbf{i}}m/s(25\,\text{s}) + 1\,\hat{\mathbf{j}}m/s(25\,\text{s})$$
$$+1/2(0.80\,\hat{\mathbf{i}}\text{m}/s^2)(25\,\text{s})^2 + 1/2(-0.30\,\hat{\mathbf{j}}\text{m}/s^2)(25\,\text{s})^2$$
$$\vec{r(t)} = 360\,\hat{\mathbf{i}}m - 72.75\,\hat{\mathbf{j}}m$$

In order to determine the direction of the motion we have to consider the final velocity vector at $t = 25\,\text{s}$. The components of the final velocity vector at $t = 25\,\text{s}$ are then:

$$\vec{v}(t) = \vec{v}_0 + \vec{a}t$$
$$\vec{v}(t) = 4\,\hat{\mathbf{i}}\,\frac{\text{m}}{\text{s}} + 1\,\hat{\mathbf{j}}\,\frac{\text{m}}{\text{s}} + 0.80\,\hat{\mathbf{i}}\,\frac{\text{m}}{\text{s}^2}(25\,\text{s}) - 0.30\,\hat{\mathbf{j}}\,\frac{\text{m}}{\text{s}^2}(25\,\text{s})$$
$$\vec{v}(t) = 24\,\hat{\mathbf{i}}\,\frac{\text{m}}{\text{s}} - 6.50\,\hat{\mathbf{j}}\,\frac{\text{m}}{\text{s}}$$

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Then we get for the direction

$$\theta = \tan^{-1}\left(\frac{v_y}{v_x}\right) = \tan^{-1}\left(\frac{-6.50\,\frac{\mathrm{m}}{\mathrm{s}}}{24\,\frac{\mathrm{m}}{\mathrm{s}}}\right) = -15.15^\circ$$

$$\theta_4 = -15.15^\circ + 360^\circ = 344.90^\circ$$

Which corresponds to angle of 344.90° with respect to the positive x-direction.

3.

(a) As the position vector is given as a function of time we can calculate the velocity easily by

$$\vec{v}(t) = \frac{d\vec{r}(t)}{dt} = \frac{d}{dt}\left(3t\hat{\mathbf{i}} - 4t^2\hat{\mathbf{j}} + 2\hat{\mathbf{k}}\right) = 3\,\frac{\mathrm{m}}{\mathrm{s}}\hat{\mathbf{i}} - 8\,\frac{\mathrm{m}}{\mathrm{s}^2}t\hat{\mathbf{j}}$$

(b) Evaluating this result at $t = 2s$ produces

$$\vec{v}(t = 2.00\,\mathrm{s}) = 3\hat{\mathbf{i}}\,\frac{\mathrm{m}}{\mathrm{s}} - 8\hat{\mathbf{j}}\,\frac{\mathrm{m}}{\mathrm{s}^2}(2\,\mathrm{s}) = 3\hat{\mathbf{i}}\,\frac{\mathrm{m}}{\mathrm{s}} - 16\hat{\mathbf{j}}\,\frac{\mathrm{m}}{\mathrm{s}}$$

(c) The speed at $t = 2\,\mathrm{s}$ is

$$v = \sqrt{\left(3\,\frac{\mathrm{m}}{\mathrm{s}}\right)^2 + \left(-16\,\frac{\mathrm{m}}{\mathrm{s}}\right)^2} = 16.28\,\frac{\mathrm{m}}{\mathrm{s}}$$

(d) The angle of $\vec{v}$ relative to the positive direction of the x axis is

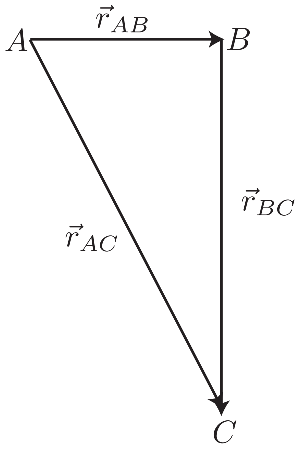
$$\theta = \tan^{-1}\left(\frac{-16\,\frac{\mathrm{m}}{\mathrm{s}}}{3\,\frac{\mathrm{m}}{\mathrm{s}}}\right) = -79.34^\circ$$

.

$$\theta_4 = -79.34^\circ + 360^\circ = 280.63^\circ$$

Which corresponds to the angle of 280.63° with respect to the positive direction of the x-axis.

4. Before we start answering the question let us draw a picture for the situation described in the question



(a) First let us write down the displacements for the two parts of the trip explicitly:

$$\vec{r}_{AB} = 483\,\hat{\mathbf{i}}\,\text{km}$$
$$\vec{r}_{BC} = -966\,\hat{\mathbf{j}}\,\text{km}$$

Then we get for the displacement $\vec{r}_{AC}$:

$$\vec{r}_{AC} = \vec{r}_{AB} + \vec{r}_{BC} = 483\,\hat{\mathbf{i}}\,\text{km} - 966\,\hat{\mathbf{j}}\,\text{km}$$

Therefore we get for the magnitude of the displacement, i.e. the trip was conducted directly from city$A$ to city C.

$$|\vec{r}_{AC}| = r_{AC} = \sqrt{(483\,\text{km})^2 + (-966\,\text{km})^2} = 1.08 \times 10^3\,\text{km}$$

(b) The direction is then given as:

$$\theta = \tan^{-1}\left(\frac{-966\,\text{km}}{483\,\text{km}}\right) = -63.43^\circ$$
$$\theta_4 = -63.43^\circ + 360^\circ = 296.57^\circ$$

Which corresponds to the angle of 296.57° with respect to the positive direction of the x-axis.

(c) For the average velocity we can use the displacement vector over the time interval.

$$\vec{v}_{av} = \frac{\vec{r}_{AC}}{\Delta t} = \frac{483\,\hat{\mathbf{i}}\,\text{km} - 966\,\hat{\mathbf{j}}\,\text{km}}{45\,\text{min}\frac{1\,\text{h}}{60\,\text{min}} + 1.5\,\text{h}} = \frac{483\,\hat{\mathbf{i}}\,\text{km} - 966\,\hat{\mathbf{j}}\,\text{km}}{2.25\,\text{h}} = 214.67\,\hat{\mathbf{i}}\,\frac{\text{km}}{\text{h}} - 429.33\,\hat{\mathbf{j}}\,\frac{\text{km}}{\text{h}}$$

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So the magnitude of the average velocity is then

$$|\vec{v}_{av}| = v_{av} = \sqrt{\left(214.67\,\frac{\text{km}}{\text{h}}\right)^2 + \left(-429.33\,\frac{\text{km}}{\text{h}}\right)^2} = 480\,\frac{\text{km}}{\text{h}}.$$

(d) The direction of the average velocity is parallel to the displacement vector and therefore 296.57° with respect to the positive direction of the x-axis or 63.4° south of west.

(e) Since the average speed is the total distance divided by the total time, we get for $v_{av.speed}$

$$v_{av.speed} = \frac{483\,\text{km} + 966\,\text{km}}{2.25\,\text{h}} = 644\,\frac{\text{km}}{\text{h}}$$

5.

(a) For calculating the position vectors in specific time we have to substitute specific time in $r\vec{(t)} = (2t^3 - 5t)\hat{\mathbf{i}} + (6 - 7t^4)\hat{\mathbf{j}}$ as you can see in below:

$$\vec{r}(t=0) = (2(0)^3 - 5(0))\hat{\mathbf{i}}m + (6 - 7(0)^4)\hat{\mathbf{j}}m = 6\hat{\mathbf{j}}m$$
$$\vec{r}(t=1s) = (2(1)^3 - 5(1))\hat{\mathbf{i}}m + (6 - 7(1)^4)\hat{\mathbf{j}}m = -3\hat{\mathbf{i}}m - 1\hat{\mathbf{j}}m$$

(b) For calculating the displacement:

$$\Delta\vec{r} = \vec{r}(t=1s) - \vec{r}(t=0s) = -3\hat{\mathbf{i}}m - 1\hat{\mathbf{j}}m - 6\hat{\mathbf{j}}m$$
$$\Delta\vec{r} = -3\hat{\mathbf{i}}m - 7\hat{\mathbf{j}}m$$

(c) For calculating the average velocity:

$$\vec{v}(t)_{avg} = \frac{\Delta\vec{r}}{\Delta t} = \frac{-3\hat{\mathbf{i}}m - 7\hat{\mathbf{j}}m}{1s - 0} == -3\hat{\mathbf{i}}\frac{m}{s} - 7\hat{\mathbf{j}}\frac{m}{s}$$

(d) For calculating the instantaneous velocity, we have to take a derivative of position vector with respect to the time(t)

$$\vec{v}(t) = \frac{d\vec{r}(t)}{dt} = (6t^2 - 5)\frac{m}{s}\hat{\mathbf{i}} - 28t^3\frac{m}{s}\hat{\mathbf{j}}$$

So the instantaneous velocity in $t = 1s$ is:

$$\vec{v}(t=1s) = 1\frac{m}{s}\hat{\mathbf{i}} - 28\frac{m}{s}\hat{\mathbf{j}}$$

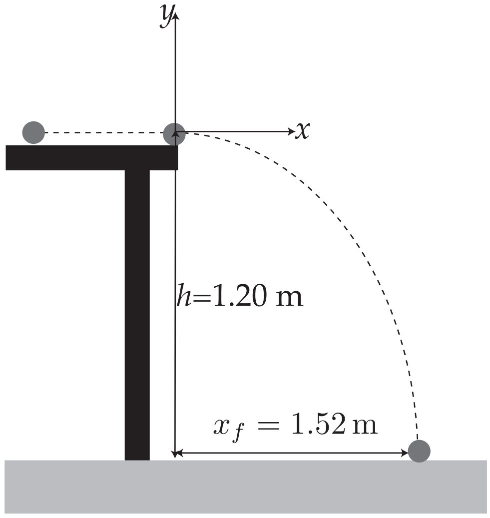
For calculating the instantaneous acceleration, we have to take a derivative of instantaneous velocity with respect to the time(t)

$$\vec{a}(t) = \frac{d\vec{v}(t)}{dt} = 12t\hat{\mathbf{i}}\frac{m}{s} - 84t^2\hat{\mathbf{j}}\frac{m}{s}$$

and the instantaneous acceleration in $t = 1s$ is:

$$\vec{a}(t=1s) = 12\hat{\mathbf{i}}\frac{m}{s} - 84\hat{\mathbf{j}}\frac{m}{s}$$

6. First let us draw a picture to understand the situation:



For calculating the duration and speed of the ball we need to substitute our data in$\vec{r(t)} = \vec{r}_0 + \vec{v}_0 t + \frac{1}{2}\vec{a}t^2$. As the ball is rolling horizontally, there is no vertical component of the initial velocity in y-direction. As illustrated in the figure above we put the origin of the coordinate system to the edge of the table, with the y-axis pointing upward and the x-axis pointing to the right.

$$\vec{r(t)} = 1.52\,\hat{\mathbf{i}}m - 0\,\hat{\mathbf{j}}, \qquad \vec{r_0} = 0\,\hat{\mathbf{i}} - 1.20\,\hat{\mathbf{j}}m.$$

$$\vec{v_0} = \vec{v_{0x}}\,\hat{\mathbf{i}} + 0\,\hat{\mathbf{j}}, \qquad \vec{a} = 0\,\hat{\mathbf{i}} - g\,\hat{\mathbf{j}}.$$

with substituting these data in general formula:

$$1.52\,\hat{\mathbf{i}}m - 0\,\hat{\mathbf{j}} = 0\,\hat{\mathbf{i}} - 1.20\,\hat{\mathbf{j}}m + \vec{v_{0x}}\,\hat{\mathbf{i}}\frac{m}{s}t + 0\,\hat{\mathbf{j}} + 0\,\hat{\mathbf{i}} - \frac{1}{2}gt^2\,\hat{\mathbf{j}}$$

(a) With matching for all y-components, we can find duration of the ball:

$$0 = -1.20 + 4.9t^2 \Longleftrightarrow t = \sqrt{\frac{1.20}{4.9}}s = 0.50s$$

(b) With matching for all x-components, and $t = 0.50s$we can find initial speed of the ball:

$$1.52m = \vec{v_{0x}}\frac{m}{s}t \Longleftrightarrow \vec{v_{0x}} = \frac{1.52}{0.50}m/s = 3.04m/s$$

7.

(a) As depicted in the figure above we put the origin of the coordinate system to the point where the stone is thrown. The direction of the positive y axis is pointing upward, and the direction of the positive x-axis is pointing to the right.

$$\vec{r}(t) = \vec{r}_0 + \vec{v}_0 t a \frac{1}{2} a t^2$$

$$R\,\hat{\mathbf{i}} + h\,\hat{\mathbf{j}} = 42\cos(60^\circ)\,5.50\,\mathrm{s}\,\hat{\mathbf{i}}\,\frac{\mathrm{m}}{\mathrm{s}} + 42\sin(60^\circ)\,5.50\,\mathrm{s}\,\hat{\mathbf{j}}\,\frac{\mathrm{m}}{\mathrm{s}} - \frac{1}{2}9.81(5.50\,\mathrm{s})^2\,\hat{\mathbf{j}}\,\frac{\mathrm{m}}{\mathrm{s}^2}$$

$$R\,\hat{\mathbf{i}} + h\,\hat{\mathbf{j}} = 115.50\,\hat{\mathbf{i}}\,\mathrm{m} + 200.03\,\hat{\mathbf{j}}\,\mathrm{m} - 148.23\,\hat{\mathbf{j}}\,\mathrm{m}$$

With matching of y-components we are be able to find height of the cliff:

$$h = 51.78\,\mathrm{m}$$

(b) Before we determine the speed, let us determine the velocity in component notation at the moment of the impact.

$$\vec{v}(t) = \vec{v}_0 + \vec{a}_0 t \Longrightarrow v_x(t)\,\hat{\mathbf{i}} + v_y(t)\,\hat{\mathbf{j}} = v_{x_0}\,\hat{\mathbf{i}} + v_{x_0}\,\hat{\mathbf{j}} - gt\,\hat{\mathbf{j}}$$

With matching of x-components and$y$-components we are be able to find the speed of the cliff before impact at A:

$$v_x(t) = v_{x_0} = 42\,\frac{\mathrm{m}}{\mathrm{s}}\cos(60^\circ) = 21\,\frac{\mathrm{m}}{\mathrm{s}}$$

$$v_y(t) = v_{y_0} - gt = 42\,\frac{\mathrm{m}}{\mathrm{s}}\sin(60^\circ) - 9.81\,\frac{\mathrm{m}}{\mathrm{s}^2}5.50\,\mathrm{s} = -17.58\,\frac{\mathrm{m}}{\mathrm{s}}$$

Then we get the speed as the magnitude of the velocity vector as

$$v = \sqrt{v_x(t)^2 + v_y(t)^2} = \sqrt{\left(21\,\frac{\mathrm{m}}{\mathrm{s}}\right)^2 + \left(-17.58\,\frac{\mathrm{m}}{\mathrm{s}}\right)^2} = 27.39\,\frac{\mathrm{m}}{\mathrm{s}}$$

(c) At the highest point of the trajectory we have $v_y = 0$ and $y = H$:

$$\vec{r}(t) = \vec{r}_0 + \vec{v}_0 t a \frac{1}{2} a t^2$$

$$\frac{R}{2}\,\hat{\mathbf{i}} + H\,\hat{\mathbf{j}} = 42\cos(60^\circ)\,5.50\,\mathrm{s}\,\hat{\mathbf{i}}\,\frac{\mathrm{m}}{\mathrm{s}} + 42\sin(60^\circ)\,5.50\,\mathrm{s}\,\hat{\mathbf{j}}\,\frac{\mathrm{m}}{\mathrm{s}} - \frac{1}{2}9.81(5.50\,\mathrm{s})^2\,\hat{\mathbf{j}}\,\frac{\mathrm{m}}{\mathrm{s}^2}$$

With matching of y-components we are be able to find the maximum height of the cliff:

$$H = 42\,\frac{\mathrm{m}}{\mathrm{s}}\sin(60^\circ)\,5.50\,\mathrm{s}\,\hat{\mathbf{j}} - \frac{1}{2}9.81\,\frac{\mathrm{m}}{\mathrm{s}^2}(5.50\,\mathrm{s})^2\,\hat{\mathbf{j}}$$

$$H = 51.67m$$

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