Jacobian

EENG428 Introduction to Robotics

Differential Motions of a Robot and its Hand Frame

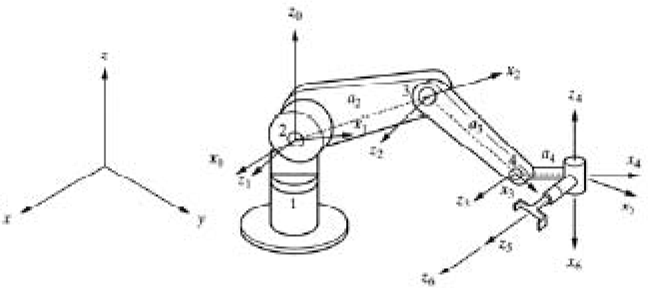
- Differential Motions
  - $dT = \Delta T = T^T\Delta$: changes in the components of n,o,a,p
  - Jacobian relates the joint micro-movements to the hand micro-movement.

$$\begin{bmatrix} dx \\ dy \\ dz \\ \delta x \\ \delta y \\ \delta z \end{bmatrix} = \begin{bmatrix} \\ \\ \text{Robot} \\ \text{Jacobian Matrix} \\ \\ \\ \end{bmatrix} \begin{bmatrix} d\theta_1 \\ d\theta_2 \\ d\theta_3 \\ d\theta_4 \\ d\theta_5 \\ d\theta_6 \end{bmatrix} \quad \text{or} \quad [D]=[J][D_\theta]$$

Calculation of the Jacobian

- $[D] = [J]\,[D_\theta]$
- In Ex 2.19), 1~3 rows of J :
  - dx, dy, dz : translation in T
  - the last column is $\{p_x, p_y, p_z, 1\}$

    $dp_x \rightarrow dx$, $dp_y \rightarrow dy$, $dp_z \rightarrow dz$



$$^{R}T_H = A_1A_2A_3A_4A_5A_6$$

$$= \begin{bmatrix} c_1(c_{234}c_5c_6 - s_{234}s_6) - s_1s_5c_6 & c_1(-c_{234}c_5c_6 - s_{234}s_6) + s_1s_5s_6 & c_1(c_{234}s_5) + s_1c_5 & c_1(c_{234}a_4 + c_{23}a_3 + c_2a_2) \\ s_1(c_{234}c_5c_6 - s_{234}s_6) + c_1s_5s_6 & s_1(-c_{234}c_5c_6 - s_{234}s_6) - c_1s_5s_6 & s_1(c_{234}s_5) - c_1c_5 & s_1(c_{234}a_4 + c_{23}a_3 + c_2a_2) \\ s_{234}c_5c_6 + c_{234}s_6 & -s_{234}c_5c_6 + c_{234}c_6 & s_{234}s_5 & s_{234}a_4 + s_{23}a_3 + s_2a_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

D−H parameters

#	Θ	d	a	α
1	Θ_1	0	0	90
2	Θ_2	0	a_2	0
3	Θ_3	0	a_3	0
4	Θ_4	0	a_4	−90
5	Θ_5	0	0	90
6	Θ_6	0	0	0

Calculation of the Jacobian

$$\begin{bmatrix} p_x \\ p_y \\ p_z \\ 1 \end{bmatrix} = \begin{bmatrix} c_1(c_{234}a_4 + c_{23}a_3 + c_2a_2) \\ s_1(c_{234}a_4 + c_{23}a_3 + c_2a_2) \\ s_{234}a_4 + s_{23}a_3 + s_2a_2 \\ 1 \end{bmatrix}$$

- $dp_x = dx$: use the chain rule

$$p_x = c_1(c_{234}a_4 + c_{23}a_3 + c_2a_2)$$

$$\begin{aligned} dp_x &= \frac{\partial p_x}{\partial \theta_1} d\theta_1 + \frac{\partial p_x}{\partial \theta_2} d\theta_2 + \cdots + \frac{\partial p_x}{\partial \theta_6} d\theta_6 \\ &= -s_1(c_{234}a_4 + c_{23}a_3 + c_2a_2) d\theta_1 + c_1[-s_{234}a_4 - s_{23}a_3 - s_2a_2] d\theta_2 \\ &\quad + c_1[-s_{234}a_4 - s_{23}a_3] d\theta_3 + c_1[-s_{234}a_4] d\theta_4 \end{aligned}$$

$\therefore$ the 1st row of Jacobian :

$$\frac{\partial p_x}{\partial \theta_1} = J_{11} = -s_1(c_{234}a_4 + c_{23}a_3 + c_2a_2) \qquad \frac{\partial p_x}{\partial \theta_4} = J_{14} = c_1[-s_{234}a_4]$$

$$\frac{\partial p_x}{\partial \theta_2} = J_{12} = c_1[-s_{234}a_4 - s_{23}a_3 - s_2a_2] \qquad \frac{\partial p_x}{\partial \theta_5} = J_{15} = 0$$

$$\frac{\partial p_x}{\partial \theta_3} = J_{13} = c_1[-s_{234}a_4 - s_{23}a_3] \qquad \frac{\partial p_x}{\partial \theta_6} = J_{16} = 0$$

Calculation of the Jacobian

- $dp_y = dy$: use the chain rule

$$\begin{bmatrix} p_x \\ p_y \\ p_z \\ 1 \end{bmatrix} = \begin{bmatrix} c_1(c_{234}a_4 + c_{23}a_3 + c_2a_2) \\ s_1(c_{234}a_4 + c_{23}a_3 + c_2a_2) \\ s_{234}a_4 + s_{23}a_3 + s_2a_2 \\ 1 \end{bmatrix}$$

$$p_y = s_1(c_{234}a_4 + c_{23}a_3 + c_2a_2)$$

$$dp_y = \frac{\partial p_y}{\partial \theta_1}d\theta_1 + \frac{\partial p_y}{\partial \theta_2}d\theta_2 + \cdots + \frac{\partial p_y}{\partial \theta_6}d\theta_6$$

$$= c_1(c_{234}a_4 + c_{23}a_3 + c_2a_2)d\theta_1 + s_1[-s_{234}a_4 - s_{23}a_3 - s_2a_2]d\theta_2$$

$$+ s_1[-s_{234}a_4 - s_{23}a_3]d\theta_3 + s_1[-s_{234}a_4]d\theta_4$$

$\therefore$ the 2^{nd} row of Jacobian :

$$\frac{\partial p_y}{\partial \theta_1} = J_{21} = c_1(c_{234}a_4 + c_{23}a_3 + c_2a_2) \qquad \frac{\partial p_y}{\partial \theta_4} = J_{24} = s_1[-s_{234}a_4]$$

$$\frac{\partial p_y}{\partial \theta_2} = J_{22} = s_1[-s_{234}a_4 - s_{23}a_3 - s_2a_2] \qquad \frac{\partial p_y}{\partial \theta_5} = J_{25} = 0$$

$$\frac{\partial p_y}{\partial \theta_3} = J_{23} = s_1[-s_{234}a_4 - s_{23}a_3] \qquad \frac{\partial p_y}{\partial \theta_6} = J_{26} = 0$$

Calculation of the Jacobian

- $dp_z = dz$: use the chain rule

$$\begin{bmatrix} p_x \\ p_y \\ p_z \\ 1 \end{bmatrix} = \begin{bmatrix} c_1(c_{234}a_4 + c_{23}a_3 + c_2a_2) \\ s_1(c_{234}a_4 + c_{23}a_3 + c_2a_2) \\ s_{234}a_4 + s_{23}a_3 + s_2a_2 \\ 1 \end{bmatrix}$$

$$p_z = s_{234}a_4 + s_{23}a_3 + s_2a_2$$

$$dp_z = \frac{\partial p_z}{\partial \theta_1} d\theta_1 + \frac{\partial p_z}{\partial \theta_2} d\theta_2 + \cdots + \frac{\partial p_z}{\partial \theta_6} d\theta_6$$

$$= [c_{234}a_4 + c_{23}a_3 + c_2a_2] d\theta_2 + [c_{234}a_4 + c_{23}a_3] d\theta_3 + [c_{234}a_4] d\theta_4$$

$\therefore$ the 3rd row of Jacobian :

$$\frac{\partial p_z}{\partial \theta_1} = J_{31} = 0$$

$$\frac{\partial p_z}{\partial \theta_2} = J_{32} = [c_{234}a_4 + c_{23}a_3 + c_2a_2]$$

$$\frac{\partial p_z}{\partial \theta_3} = J_{33} = [c_{234}a_4 + c_{23}a_3]$$

$$\frac{\partial p_z}{\partial \theta_4} = J_{34} = [c_{234}a_4]$$

$$\frac{\partial p_z}{\partial \theta_5} = J_{35} = 0$$

$$\frac{\partial p_z}{\partial \theta_6} = J_{36} = 0$$

Calculation of the Jacobian

- In Ex 2.19), 4~6 rows of J :
  - δx, δy, δz : rotation in T
  - But, no unique eq. that describes the rotations about the 3 axes. So, there is no single eq. for differential rotations about the 3 axes. → different calculation needed
- Simple Formula [Paul] :
  - The Jacobian w.r.t. T_6 (the last frame) is simpler to compute than that w.r.t. the first frame. [Paul]

$$\left[{}^{T_6}D \right] = \left[{}^{T_6}J \right]\left[D_\theta \right]$$

  - → for the same joint differential motions, premultiplied with the Jacobian w.r.t. the last frame, the hand differential motions w.r.t. the last frame is obtained.

Calculation of the Jacobian

- Simple Formula [Paul]

$$\begin{bmatrix} {}^{T_6}dx \\ {}^{T_6}dy \\ {}^{T_6}dz \\ {}^{T_6}\delta x \\ {}^{T_6}\delta y \\ {}^{T_6}\delta z \end{bmatrix} = \begin{bmatrix} {}^{T_6}J_{11} & {}^{T_6}J_{12} & \cdots & {}^{T_6}J_{16} \\ {}^{T_6}J_{21} & {}^{T_6}J_{22} & \cdots & {}^{T_6}J_{26} \\ {}^{T_6}J_{31} & . & \cdots & {}^{T_6}J_{36} \\ {}^{T_6}J_{41} & . & \cdots & {}^{T_6}J_{46} \\ {}^{T_6}J_{51} & . & \cdots & {}^{T_6}J_{56} \\ {}^{T_6}J_{61} & . & \cdots & {}^{T_6}J_{66} \end{bmatrix} \begin{bmatrix} d\theta_1 \\ d\theta_2 \\ d\theta_3 \\ d\theta_4 \\ d\theta_5 \\ d\theta_6 \end{bmatrix}$$

Use ${}^{i-1}T_6$ for the ith column of ${}^{T_6}J$ $\left({}^{T_6}J_{ki}\right)$

1st column : ${}^{0}T_6 = A_1A_2A_3A_4A_5A_6$

2nd column : ${}^{1}T_6 = A_2A_3A_4A_5A_6$

3rd column : ${}^{2}T_6 = A_3A_4A_5A_6$

4th column : ${}^{3}T_6 = A_4A_5A_6$

5th column : ${}^{4}T_6 = A_5A_6$

6th column : ${}^{5}T_6 = A_6$

- {n,o,a,p} matrix represented by any combination of $A_1 \sim A_6$ is used to calculate the Jacobian. (e.g., ${}^{0}T_6 \sim {}^{5}T_6$)
- ith revolute joint : ${}^{T_6}J_{1i} = \left(-n_x p_y + n_y p_x\right)$ ${}^{T_6}J_{2i} = \left(-o_x p_y + o_y p_x\right)$ ${}^{T_6}J_{3i} = \left(-a_x p_y + a_y p_x\right)$

  ${}^{T_6}J_{4i} = n_z$ ${}^{T_6}J_{5i} = o_z$ ${}^{T_6}J_{6i} = a_z$
- ith prismatic joint : ${}^{T_6}J_{1i} = n_z$ ${}^{T_6}J_{2i} = o_z$ ${}^{T_6}J_{3i} = a_z$

  ${}^{T_6}J_{4i} = 0$ ${}^{T_6}J_{5i} = 0$ ${}^{T_6}J_{6i} = 0$

Calculation of the Jacobian

- Ex 3.9) Find the ${}^{T6}J_{11}$ and ${}^{T6}J_{41}$ elements of the Jacobian for the simple revolute robot.

$${}^{R}T_{H} = A_1A_2A_3A_4A_5A_6$$

$$= \begin{bmatrix} c_1(c_{234}c_5c_6 - s_{234}s_6) - s_1s_5c_6 & c_1(-c_{234}c_5c_6 - s_{234}s_6) + s_1s_5s_6 & c_1(c_{234}s_5) + s_1c_5 & c_1(c_{234}a_4 + c_{23}a_3 + c_2a_2) \\ s_1(c_{234}c_5c_6 - s_{234}s_6) + c_1s_5s_6 & s_1(-c_{234}c_5c_6 - s_{234}s_6) - c_1s_5s_6 & s_1(c_{234}s_5) - c_1c_5 & s_1(c_{234}a_4 + c_{23}a_3 + c_2a_2) \\ s_{234}c_5c_6 + c_{234}s_6 & -s_{234}c_5c_6 + c_{234}c_6 & s_{234}s_5 & s_{234}a_4 + s_{23}a_3 + s_2a_2 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} n_x & o_x & a_x & p_x \\ n_y & o_y & a_y & p_y \\ n_z & o_z & a_z & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

For the i^th^ revolute joint, by eq(3.25)

$${}^{T_6}J_{1i} = (-n_xp_y + n_yp_x) \qquad {}^{T_6}J_{2i} = (-o_xp_y + o_yp_x) \qquad {}^{T_6}J_{3i} = (-a_xp_y + a_yp_x)$$

$${}^{T_6}J_{4i} = n_z \qquad {}^{T_6}J_{5i} = o_z \qquad {}^{T_6}J_{6i} = a_z$$

Substituting $i = 1$,

$${}^{T_6}J_{11} = (-n_xp_y + n_yp_x) = -[c_1(c_{234}c_5c_6 - s_{234}s_6) - s_1s_5c_6] \times [s_1(c_{234}a_4 + c_{23}a_3 + c_2a_2)]$$

$$+ [s_1(c_{234}c_5c_6 - s_{234}s_6) + c_1s_5s_6] \times [c_1(c_{234}a_4 + c_{23}a_3 + c_2a_2)]$$

$$= s_5c_6(c_{234}a_4 + c_{23}a_3 + c_2a_2)$$

$${}^{T_6}J_{41} = n_z = s_{234}c_5c_6 + c_{234}s_6$$

Calculation of the Jacobian

- → Compare eq(3.23) with eq(3.28) with respect to J_{11}.

$$J_{11} = -s_1(c_{234}a_4 + c_{23}a_3 + c_2a_2)$$
$$J_{12} = c_1[-s_{234}a_4 - s_{23}a_3 - s_2a_2]$$
$$J_{13} = c_1[-s_{234}a_4 - s_{23}a_3]$$
$$J_{14} = c_1[-s_{234}a_4]$$
$$J_{16} = 0$$
$$J_{15} = 0$$

$${}^{T_6}J_{11} = (-n_xp_y + n_yp_x) = s_5c_6(c_{234}a_4 + c_{23}a_3 + c_2a_2)$$
$${}^{T_6}J_{41} = n_z = s_{234}c_5c_6 + c_{234}s_6$$

J_{11} w.r.t. the reference frame

J_{11} w.r.t. the current or T_6 frame

Calculation of the Jacobian

SCARA Manipulator

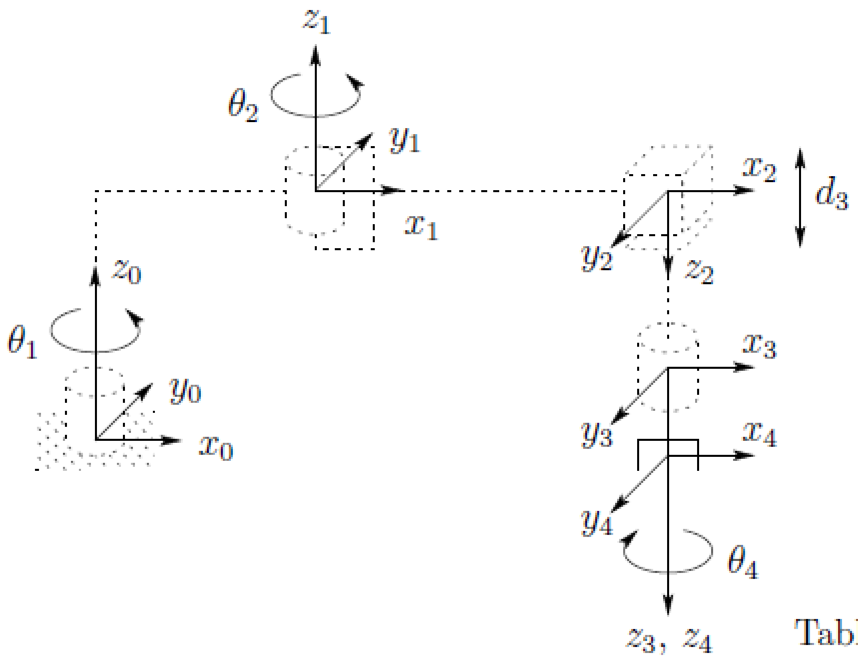


Table 3.5: Joint parameters for SCARA.

Link	a_i	α_i	d_i	θ_i
1	a_1	0	0	$\star$
2	a_2	180	0	$\star$
3	0	0	$\star$	0
4	0	0	d_4	$\star$

* joint variable

Jacobian of SCARA Manipulator

$$
\begin{aligned}
A_1 &= \begin{bmatrix} c_1 & -s_1 & 0 & a_1c_1 \\ s_1 & c_1 & 0 & a_1s_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\
A_2 &= \begin{bmatrix} c_2 & s_2 & 0 & a_2c_2 \\ s_2 & -c_2 & 0 & a_2s_2 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\
A_3 &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\
A_4 &= \begin{bmatrix} c_4 & -s_4 & 0 & 0 \\ s_4 & c_4 & 0 & 0 \\ 0 & 0 & 1 & d_4 \\ 0 & 0 & 0 & 1 \end{bmatrix}.
\end{aligned}
$$

Jacobian of SCARA Manipulator

$$A_3A_4 = \begin{bmatrix} C_4 & -S_4 & 0 & 0 \\ S_4 & C_4 & 0 & 0 \\ 0 & 0 & 1 & d_3+d_4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A_2A_3A_4 = \begin{bmatrix} C_2C_4+S_2S_4 & S_2C_4-C_2S_4 & 0 & a_2C_2 \\ S_2C_4-C_2S_4 & -C_2C_4-S_2S_4 & 0 & a_2S_2 \\ 0 & 0 & -1 & -d_3-d_4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A_1A_2A_3A_4 = \begin{bmatrix} C_{12}C_4+S_{12}S_4 & S_{12}C_4-C_{12}S_4 & 0 & a_1C_1+a_2C_{12} \\ S_{12}C_4-C_{12}S_4 & -C_{12}C_4-S_{12}S_4 & 0 & a_1S_1+a_2S_{12} \\ 0 & 0 & -1 & -d_3-d_4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Jacobian of SCARA Manipulator

- i^{th} revolute joint : ${}^{T_6}J_{1i} = (-n_x p_y + n_y p_x)$ ${}^{T_6}J_{2i} = (-o_x p_y + o_y p_x)$ ${}^{T_6}J_{3i} = (-a_x p_y + a_y p_x)$
  ${}^{T_6}J_{4i} = n_z$ ${}^{T_6}J_{5i} = o_z$ ${}^{T_6}J_{6i} = a_z$
- i^{th} prismatic joint : ${}^{T_6}J_{1i} = n_z$ ${}^{T_6}J_{2i} = o_z$ ${}^{T_6}J_{3i} = a_z$
  ${}^{T_6}J_{4i} = 0$ ${}^{T_6}J_{5i} = 0$ ${}^{T_6}J_{6i} = 0$

Since the first joint is revolute: (Consider $A_1A_2A_3A_4$)

$$J_{11} = -(C_{12}C_4 + S_{12}S_4)(a_1S_1 + a_2S_{12}) + (a_1C_1 + a_2C_{12})(S_{12}C_4 - C_{12}S_4)$$
$$J_{11} = -C(\theta_1 + \theta_2 - \theta_4)(a_1S_1 + a_2S_{12}) + S(\theta_1 + \theta_2 - \theta_4)(a_1C_1 + a_2C_{12})$$
$$J_{11} = a_1\left[S(\theta_1 + \theta_2 - \theta_4)C_1 - S_1C(\theta_1 + \theta_2 - \theta_4)\right] + a_2\left[S(\theta_1 + \theta_2 - \theta_4)C_{12} - S_{12}C(\theta_1 + \theta_2 - \theta_4)\right]$$
$$J_{11} = a_1S(\theta_2 - \theta_4) - a_2S_4$$

$$J_{12} = -S(\theta_1 + \theta_2 - \theta_4)(a_1S_1 + a_2S_{12}) - C(\theta_1 + \theta_2 - \theta_4)(a_1C_1 + a_2C_{12})$$
$$J_{12} = -a_1\left[S(\theta_1 + \theta_2 - \theta_4)S_1 + C_1C(\theta_1 + \theta_2 - \theta_4)\right] - a_2\left[S(\theta_1 + \theta_2 - \theta_4)S_{12} + C_{12}C(\theta_1 + \theta_2 - \theta_4)\right]$$
$$J_{12} = -a_1C(\theta_2 - \theta_4) - a_2C_4$$

Jacobian of SCARA Manipulator

- i^{th} revolute joint : ${}^{T_6}J_{1i} = \left(-n_x p_y + n_y p_x\right)$ ${}^{T_6}J_{2i} = \left(-o_x p_y + o_y p_x\right)$ ${}^{T_6}J_{3i} = \left(-a_x p_y + a_y p_x\right)$
  ${}^{T_6}J_{4i} = n_z$ ${}^{T_6}J_{5i} = o_z$ ${}^{T_6}J_{6i} = a_z$
- i^{th} prismatic joint : ${}^{T_6}J_{1i} = n_z$ ${}^{T_6}J_{2i} = o_z$ ${}^{T_6}J_{3i} = a_z$
  ${}^{T_6}J_{4i} = 0$ ${}^{T_6}J_{5i} = 0$ ${}^{T_6}J_{6i} = 0$

Since the first joint is revolute:

$$J_{31} = 0$$

$$J_{41} = 0$$

$$J_{51} = 0$$

$$J_{61} = -1$$

Jacobian of SCARA Manipulator

- i^{th} revolute joint : ${}^{T_6}J_{1i} = (-n_x p_y + n_y p_x)$ ${}^{T_6}J_{2i} = (-o_x p_y + o_y p_x)$ ${}^{T_6}J_{3i} = (-a_x p_y + a_y p_x)$
  ${}^{T_6}J_{4i} = n_z$ ${}^{T_6}J_{5i} = o_z$ ${}^{T_6}J_{6i} = a_z$
- i^{th} prismatic joint : ${}^{T_6}J_{1i} = n_z$ ${}^{T_6}J_{2i} = o_z$ ${}^{T_6}J_{3i} = a_z$
  ${}^{T_6}J_{4i} = 0$ ${}^{T_6}J_{5i} = 0$ ${}^{T_6}J_{6i} = 0$

Since the second joint is revolute as well:

(Consider $A_2A_3A_4$)

$$J_{12} = -a_2 S_4$$
$$J_{22} = -a_2 C_4$$
$$J_{32} = 0$$
$$J_{42} = 0$$
$$J_{52} = 0$$
$$J_{62} = -1$$

Jacobian of SCARA Manipulator

- i^{th} revolute joint : ${}^{T_6}J_{1i} = (-n_x p_y + n_y p_x)$ ${}^{T_6}J_{2i} = (-o_x p_y + o_y p_x)$ ${}^{T_6}J_{3i} = (-a_x p_y + a_y p_x)$
  ${}^{T_6}J_{4i} = n_z$ ${}^{T_6}J_{5i} = o_z$ ${}^{T_6}J_{6i} = a_z$
- i^{th} prismatic joint : ${}^{T_6}J_{1i} = n_z$ ${}^{T_6}J_{2i} = o_z$ ${}^{T_6}J_{3i} = a_z$
  ${}^{T_6}J_{4i} = 0$ ${}^{T_6}J_{5i} = 0$ ${}^{T_6}J_{6i} = 0$

The third joint is prismatic:

(Consider A_3A_4)

$$J_{13} = 0$$
$$J_{23} = 0$$
$$J_{33} = 1$$
$$J_{43} = 0$$
$$J_{53} = 0$$
$$J_{63} = 0$$

Jacobian of SCARA Manipulator

- i^{th} revolute joint : ${}^{T_6}J_{1i} = (-n_x p_y + n_y p_x)$ ${}^{T_6}J_{2i} = (-o_x p_y + o_y p_x)$ ${}^{T_6}J_{3i} = (-a_x p_y + a_y p_x)$
  ${}^{T_6}J_{4i} = n_z$ ${}^{T_6}J_{5i} = o_z$ ${}^{T_6}J_{6i} = a_z$
- i^{th} prismatic joint : ${}^{T_6}J_{1i} = n_z$ ${}^{T_6}J_{2i} = o_z$ ${}^{T_6}J_{3i} = a_z$
  ${}^{T_6}J_{4i} = 0$ ${}^{T_6}J_{5i} = 0$ ${}^{T_6}J_{6i} = 0$

The fourth joint is revolute:

(Consider A_4)

$$J_{14} = 0$$
$$J_{24} = 0$$
$$J_{34} = 0$$
$$J_{44} = 0$$
$$J_{54} = 0$$
$$J_{64} = 1$$

Jacobian of SCARA Manipulator

Hence

$$\begin{bmatrix} {}^{T_4}dx \\ {}^{T_4}dy \\ {}^{T_4}dz \\ {}^{T_4}\delta x \\ {}^{T_4}\delta y \\ {}^{T_4}\delta z \end{bmatrix} = \begin{bmatrix} a_1 S(\theta_2 - \theta_4) - a_2 S_4 & -a_2 S_4 & 0 & 0 \\ -a_1 C(\theta_2 - \theta_4) - a_2 C_4 & -a_2 C_4 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} d\theta_1 \\ d\theta_2 \\ dd_3 \\ d\theta_4 \end{bmatrix}$$

Calculation of Jacobian: Vector Cross Product Method

(Jacobian w.r.t base coordinate frame)

- The Jacobian is given by

$$\begin{bmatrix} \mathbf{d} \\ \boldsymbol{\delta} \end{bmatrix} = \begin{bmatrix} \mathbf{J_d} \\ \mathbf{J_\delta} \end{bmatrix} d\mathbf{q}$$

where

$$\mathbf{J_d} = \begin{bmatrix} J_{d1} & J_{d2} & \cdots & J_{dn} \end{bmatrix} \quad J_{di} = \begin{cases} {}^0a_{i-1} \times \left({}^0p_n - {}^0p_{i-1} \right) & \text{for revolute joint } i \\ {}^0a_{i-1} & \text{for prismatic joint } i \end{cases}$$

$$\mathbf{J_\delta} = \begin{bmatrix} J_{\delta 1} & J_{\delta 2} & \cdots & J_{\delta n} \end{bmatrix} \quad J_{\delta i} = \begin{cases} {}^0a_{i-1} & \text{for revolute joint } i \\ 0 & \text{for prismatic joint } i \end{cases}$$

Jacobian of SCARA Manipulator

$$A_1 = \begin{bmatrix} c_1 & -s_1 & 0 & a_1c_1 \\ s_1 & c_1 & 0 & a_1s_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} c_2 & s_2 & 0 & a_2c_2 \\ s_2 & -c_2 & 0 & a_2s_2 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A_4 = \begin{bmatrix} c_4 & -s_4 & 0 & 0 \\ s_4 & c_4 & 0 & 0 \\ 0 & 0 & 1 & d_4 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Jacobian of SCARA Manipulator

$$A_1A_2 = \begin{bmatrix} C_{12} & S_{12} & 0 & a_2C_{12}+a_1C_1 \\ S_{12} & -C_{12} & 0 & a_2S_{12}+a_1S_1 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A_1A_2A_3 = \begin{bmatrix} C_{12} & S_{12} & 0 & a_2C_{12}+a_1C_1 \\ S_{12} & -C_{12} & 0 & a_2S_{12}+a_1S_1 \\ 0 & 0 & -1 & -d_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A_1A_2A_3A_4 = \begin{bmatrix} C(\theta_1+\theta_2-\theta_4) & S(\theta_1+\theta_2-\theta_4) & 0 & a_2C_{12}+a_1C_1 \\ S(\theta_1+\theta_2-\theta_4) & -C(\theta_1+\theta_2-\theta_4) & 0 & a_2S_{12}+a_1S_1 \\ 0 & 0 & -1 & -d_3-d_4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Jacobian of SCARA Manipulator

$$J_{di} = \begin{cases} {}^{0}a_{i-1} \times \left({}^{0}p_n - {}^{0}p_{i-1} \right) & \text{for revolute joint } i \\ {}^{0}a_{i-1} & \text{for prismatic joint } i \end{cases}$$

$$J_{\delta i} = \begin{cases} {}^{0}a_{i-1} & \text{for revolute joint } i \\ 0 & \text{for prismatic joint } i \end{cases}$$

First Joint is revolute

$$J_{d1} = {}^{0}a_0 \times \left({}^{0}p_4 - {}^{0}p_0 \right)$$

$$J_{\delta 1} = {}^{0}a_0$$

$${}^{0}a_0 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \qquad {}^{0}p_0 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad {}^{0}p_4 = \begin{bmatrix} a_2 C_{12} + a_1 C_1 \\ a_2 S_{12} + a_1 S_1 \\ -d_3 - d_4 \end{bmatrix}$$

Jacobian of SCARA Manipulator

$$J_{d1} = {}^{0}a_{0} \times \left({}^{0}p_{4} - {}^{0}p_{0} \right) = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} a_2 C_{12} + a_1 C_1 \\ a_2 S_{12} + a_1 S_1 \\ -d_3 - d_4 \end{bmatrix} = \begin{bmatrix} -a_2 S_{12} - a_1 S_1 \\ a_2 C_{12} + a_1 C_1 \\ 0 \end{bmatrix}$$

$$J_{\delta 1} = {}^{0}a_{0} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Jacobian of SCARA Manipulator

$$J_{di} = \begin{cases} {}^{0}a_{i-1} \times \left({}^{0}p_{n} - {}^{0}p_{i-1} \right) & \text{for revolute joint } i \\ {}^{0}a_{i-1} & \text{for prismatic joint } i \end{cases}$$

$$J_{\delta i} = \begin{cases} {}^{0}a_{i-1} & \text{for revolute joint } i \\ 0 & \text{for prismatic joint } i \end{cases}$$

Second Joint is revolute

$$J_{d2} = {}^{0}a_{1} \times \left({}^{0}p_{4} - {}^{0}p_{1} \right) = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} a_2 C_{12} \\ a_2 S_{12} \\ -d_3 - d_4 \end{bmatrix} = \begin{bmatrix} -a_2 S_{12} \\ a_2 C_{12} \\ 0 \end{bmatrix}$$

$$J_{\delta 2} = {}^{0}a_{1} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$${}^{0}a_{1} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \qquad {}^{0}p_{1} = \begin{bmatrix} a_1 C_1 \\ a_1 S_1 \\ 0 \end{bmatrix} \qquad {}^{0}p_{4} = \begin{bmatrix} a_2 C_{12} + a_1 C_1 \\ a_2 S_{12} + a_1 S_1 \\ -d_3 - d_4 \end{bmatrix}$$

Jacobian of SCARA Manipulator

$$J_{di} = \begin{cases} {}^0a_{i-1} \times \left({}^0p_n - {}^0p_{i-1} \right) & \text{for revolute joint } i \\ {}^0a_{i-1} & \text{for prismatic joint } i \end{cases}$$

$$J_{\delta i} = \begin{cases} {}^0a_{i-1} & \text{for revolute joint } i \\ 0 & \text{for prismatic joint } i \end{cases}$$

Third Joint is prismatic

$$J_{d3} = {}^0a_2 = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}$$

$$J_{\delta 2} = 0 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Jacobian of SCARA Manipulator

$$J_{di} = \begin{cases} {}^{0}a_{i-1} \times \left({}^{0}p_{n} - {}^{0}p_{i-1} \right) & \text{for revolute joint } i \\ {}^{0}a_{i-1} & \text{for prismatic joint } i \end{cases}$$

$$J_{\delta i} = \begin{cases} {}^{0}a_{i-1} & \text{for revolute joint } i \\ 0 & \text{for prismatic joint } i \end{cases}$$

forth Joint is revolute

$$J_{d4} = {}^{0}a_{3} \times \left({}^{0}p_{4} - {}^{0}p_{3} \right) = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ -d_{4} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$J_{\delta 4} = {}^{0}a_{3} = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}$$

$${}^{0}a_{3} = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix} \qquad {}^{0}p_{3} = \begin{bmatrix} a_{2}C_{12} + a_{1}C_{1} \\ a_{2}S_{12} + a_{1}S_{1} \\ -d_{3} \end{bmatrix} \qquad {}^{0}p_{4} = \begin{bmatrix} a_{2}C_{12} + a_{1}C_{1} \\ a_{2}S_{12} + a_{1}S_{1} \\ -d_{3} - d_{4} \end{bmatrix}$$

Jacobian of SCARA Manipulator

Hence

$$\begin{bmatrix} dx \\ dy \\ dz \\ \delta x \\ \delta y \\ \delta z \end{bmatrix} = \begin{bmatrix} -a_2S_{12} - a_1S_1 & -a_2S_{12} & 0 & 0 \\ a_2C_{12} + a_1C_1 & a_2C_{12} & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} d\theta_1 \\ d\theta_2 \\ dd_3 \\ d\theta_4 \end{bmatrix}$$

Relationship between the Jacobians w.r.t base and the hand frame

$$\left[\frac{\mathbf{J_d}}{\mathbf{J_\delta}}\right] = \left[\begin{array}{c|c} {}^{0}\mathbf{R}_n & \mathbf{0} \\ \hline \mathbf{0} & {}^{0}\mathbf{R}_n \end{array}\right]\left[\frac{{}^{T_n}\mathbf{J_d}}{{}^{T_n}\mathbf{J_\delta}}\right]$$