Jacobian and the Differential Operator

EENG428 Introduction to Robotics

How to relate the Jacobian and the Differential Operator

- Jacobian vs. Differential Operator
  - differential joint motions → J → D={dx, dy, dz, δx, δy, δz} → substituted in Δ → differential transformation (dT=ΔT) → *update position/orientation of robot's hand*

$$[D]=[J][D_\theta] \qquad \begin{bmatrix} & & \\ & \text{Robot} & \\ & \text{Jacobian Matrix} & \\ & & \end{bmatrix} \begin{bmatrix} d\theta_1 \\ d\theta_2 \\ d\theta_3 \\ d\theta_4 \\ d\theta_5 \\ d\theta_6 \end{bmatrix} = \begin{bmatrix} dx \\ dy \\ dz \\ \delta x \\ \delta y \\ \delta z \end{bmatrix}$$

$$\Delta = \begin{bmatrix} 0 & -\delta z & \delta y & dx \\ \delta z & 0 & -\delta x & dy \\ -\delta y & \delta x & 0 & dz \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

  - alternatively, differential joint motions → ${}^{T6}J$ → ${}^{T6}D$={${}^{T6}dx$, ${}^{T6}dy$, ${}^{T6}dz$, ${}^{T6}\delta x$, ${}^{T6}\delta y$, ${}^{T6}\delta z$} → substituted in ${}^{T}\Delta$ → differential transformation ($dT=T\,{}^{T}\Delta$) → *update position/orientation of robot's hand*

How to relate the Jacobian and the Differential Operator

- Ex 3.10) 5 d.o.f. robot ~ 2RP2R config. Jacobian for this instant, differential motions are given.

$$T_6 = \begin{bmatrix} 1 & 0 & 0 & 5 \\ 0 & 0 & -1 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad J = \begin{bmatrix} 3 & 0 & 0 & 0 & 0 \\ -2 & 0 & 1 & 0 & 0 \\ 0 & 4 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} d\theta_1 \\ d\theta_2 \\ ds_3 \\ d\theta_4 \\ d\theta_5 \end{bmatrix} = \begin{bmatrix} 0.1 \\ -0.1 \\ 0.05 \\ 0.1 \\ 0 \end{bmatrix}$$

- Sol.

$$[D] = [J][D_\theta] \quad \begin{bmatrix} 3 & 0 & 0 & 0 & 0 \\ -2 & 0 & 1 & 0 & 0 \\ 0 & 4 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0.1 \\ -0.1 \\ 0.05 \\ 0.1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0.3 \\ -0.15 \\ -0.4 \\ 0 \\ -0.1 \end{bmatrix} \rightarrow \Delta = \begin{bmatrix} 0 & 0 & -0.1 & 0.3 \\ 0 & 0 & 0 & -0.15 \\ 0.1 & 0 & 0 & -0.4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$[dT_6] = [\Delta][T_6] = \begin{bmatrix} 0 & 0 & -0.1 & 0.3 \\ 0 & 0 & 0 & -0.15 \\ 0.1 & 0 & 0 & -0.4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 5 \\ 0 & 0 & -1 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -0.1 & 0 & 0.1 \\ 0 & 0 & 0 & -0.15 \\ 0.1 & 0 & 0 & 0.1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$T_{6,new} = dT_6 + T_{6,old} = \begin{bmatrix} 0 & -0.1 & 0 & 0.1 \\ 0 & 0 & 0 & -0.15 \\ 0.1 & 0 & 0 & 0.1 \\ 0 & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 & 5 \\ 0 & 0 & -1 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -0.1 & 0 & 5.1 \\ 0 & 0 & -1 & 2.85 \\ 0.1 & 1 & 0 & 2.1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Inverse Jacobian

Inverse Jacobian

- To compute the differential velocities at the joints of the robot for a desired hand differential velocities.

$$\left.\begin{aligned}[D]&=[J][D_\theta]\\ [J^{-1}][D]&=[J^{-1}][J][D_\theta]\end{aligned}\right\}\rightarrow[D_\theta]=[J^{-1}][D]$$

$$\left.\begin{aligned}[{}^{T_6}D]&=[{}^{T_6}J][D_\theta]\\ [{}^{T_6}J^{-1}][D]&=[{}^{T_6}J^{-1}][{}^{T_6}J][D_\theta]\end{aligned}\right\}\rightarrow[D_\theta]=[{}^{T_6}J^{-1}][{}^{T_6}D]$$

- ⇨ The robot follows a desired path (inverse kinematics) on a flat plane with a constant speed (inverse Jacobian).

Inverse Jacobian

- Problem
  - To compute the inverse Jacobian in real-time ~ difficult (time consuming and computationally intensive)
- Sol. (2-ways)
  - 1) find the symbolic inverse of the Jacobian → substitute the values into inv(J) to compute the velocities
  - 2) substitute the numbers in Jacobian → invert the numerical matrix by Gaussian elimination or LU decomposition
- Simpler Sol.
  - Use the inverse kinematic eq.'s to compute joint velocities
  - $d\theta_{i=1\sim6}$ = differentiate{inverse kinematics} = J(dn,do,da,dp)

Inverse Jacobian

- Simpler Sol. (inverse kinematics → inverse Jacobian)
  - eq(2.62) : $p_x s_1 - p_y c_1 = 0 \quad \rightarrow \quad \theta_1 = \tan^{-1}\left(\frac{p_y}{p_x}\right) \text{ and } \theta_1 = \theta_1 + 180^\circ$

$$p_x s_1 = p_y c_1 \quad \rightarrow \quad \text{differentiate both sides for } d\theta_1$$

$$dp_x s_1 + p_x c_1 d\theta_1 = dp_y c_1 - p_y s_1 d\theta_1 \quad \rightarrow \quad d\theta_1 (p_x c_1 + p_y s_1) = -dp_x s_1 + dp_y c_1$$

$$\therefore \quad d\theta_1 = \frac{-dp_x s_1 + dp_y c_1}{(p_x c_1 + p_y s_1)}$$

  - eq(2.68) : $s_{234}(c_1 a_x + s_1 a_y) = c_{234} a_z \rightarrow \theta_{234} = \tan^{-1}\left(\frac{a_z}{c_1 a_x + s_1 a_y}\right) \text{ and } \theta_{234} = \theta_{234} + 180^\circ$

$$\rightarrow \quad \text{differentiate both sides}$$

$$c_{234}(d\theta_2 + d\theta_3 + d\theta_4)(c_1 a_x + s_1 a_y) + s_{234}[-s_1 a_x d\theta_1 + c_1 da_x + c_1 a_y d\theta_1 + s_1 da_y]$$

$$= s_{234}(d\theta_2 + d\theta_3 + d\theta_4) a_z + c_{234} da_z$$

$$\therefore \quad (d\theta_2 + d\theta_3 + d\theta_4) = \frac{s_{234}[-s_1 a_x d\theta_1 + c_1 da_x + c_1 a_y d\theta_1 + s_1 da_y] + c_{234} da_z}{c_{234}(c_1 a_x + s_1 a_y) + s_{234} a_z}$$

Inverse Jacobian

- Simpler Sol. (inverse kinematics → inverse Jacobian)
  - eq(2.64) : $2a_2a_3c_3 = (p_xc_1 + p_ys_1 - c_{234}a_4)^2 + (p_z - s_{234}a_4)^2 - a_2^2 - a_3^2$ $\rightarrow \theta_3 = \tan^{-1}\left(\frac{s_3}{c_3}\right)$

    $s_3 = \pm\sqrt{1 - c_3^2}$

→ differentiate both sides

$$-2a_2a_3s_3d\theta_3 = 2(p_xc_1 + p_ys_1 - c_{234}a_4)[dp_xc_1 - p_xs_1d\theta_1 + dp_ys_1 + p_yc_1d\theta_1 + s_{234}a_4(d\theta_2 + d\theta_3 + d\theta_4)]$$
$$+ 2(p_z - s_{234}a_4)[dp_z - c_{234}a_4(d\theta_2 + d\theta_3 + d\theta_4)]$$

  - eq(2.70) : $s_2[(c_3a_3 + a_2)^2 + s_3^2a_3^2] = (c_3a_3 + a_2)(p_z - s_{234}a_4) - s_3a_3(p_xc_1 + p_ys_1 - c_{234}a_4)$

→ differentiate both sides

$$c_2d\theta_2[(c_3a_3 + a_2)^2 + s_3^2a_3^2] + s_2[2(c_3a_3 + a_2)(-s_3a_3d\theta_3) + 2s_3c_3a_3^2d\theta_3]$$
$$= (-s_3a_3d\theta_3)(p_z - s_{234}a_4) + (c_3a_3 + a_2)[dp_z - c_{234}a_4(d\theta_2 + d\theta_3 + d\theta_4)]$$
$$- c_3a_3d\theta_3(p_xc_1 + p_ys_1 - c_{234}a_4) - s_3a_3[dp_xc_1 - p_xs_1d\theta_1 + dp_ys_1 + p_yc_1d\theta_1 + s_{234}a_4(d\theta_2 + d\theta_3 + d\theta_4)]$$

$$\therefore d\theta_4 = \frac{s_{234}[-s_1a_xd\theta_1 + c_1da_x + c_1a_yd\theta_1 + s_1da_y] + c_{234}da_z}{c_{234}(c_1a_x + s_1a_y) + s_{234}a_z} - d\theta_2 - d\theta_3$$

Inverse Jacobian

- Simpler Sol. (inverse kinematics → inverse Jacobian)
  - eq(2.73) : $c_5 = -c_1 a_y + s_1 a_x$

    $\rightarrow$ differentiate both sides

    $$-s_5 d\theta_5 = s_1 a_y d\theta_1 - c_1 da_y + c_1 a_x d\theta_1 + s_1 da_x$$

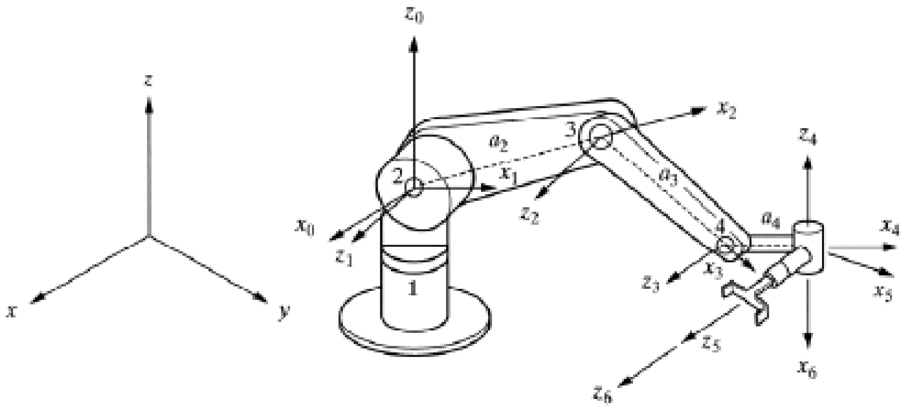
  - eq(2.75) : $s_6 = -s_{234}(c_1 n_x + s_1 n_y) + c_{234} n_z$

    $\rightarrow$ differentiate both sides

    $$c_6 d\theta_6 = -c_{234}(c_1 n_x + s_1 n_y)(d\theta_2 + d\theta_3 + d\theta_4)$$
    $$- s_{234}(-s_1 n_x d\theta_1 + c_1 dn_x + c_1 n_y d\theta_1 + s_1 dn_y)$$
    $$- s_{234} n_z (d\theta_2 + d\theta_3 + d\theta_4) + c_{234} dn_z$$

Inverse Jacobian

- Ex 3.11) Compute the angular velocity of the 1st joint s.t. the hand frame shows the followings: (inverse Jacobian)
  - dx/dt=1 in/sec, dy/dt=−2 in/sec, δx/dt=0.1 rad/sec,
  - θ_1=0, θ_2=90, θ_3=0, θ_4=90, θ_5=0, θ_6=45, a_2=a_3=a_4=15"



D-H parameters				
#	Θ	d	a	α
1	Θ_1	0	0	90
2	Θ_2	0	a_2	0
3	Θ_3	0	a_3	0
4	Θ_4	0	a_4	−90
5	Θ_5	0	0	90
6	Θ_6	0	0	0

Inverse Jacobian

$$^{R}T_H = A_1A_2A_3A_4A_5A_6$$

$$= \begin{bmatrix} c_1(c_{234}c_5c_6 - s_{234}s_6) - s_1s_5c_6 & c_1(-c_{234}c_5c_6 - s_{234}s_6) + s_1s_5s_6 & c_1(c_{234}s_5) + s_1c_5 & c_1(c_{234}a_4 + c_{23}a_3 + c_2a_2) \\ s_1(c_{234}c_5c_6 - s_{234}s_6) + c_1s_5s_6 & s_1(-c_{234}c_5c_6 - s_{234}s_6) - c_1s_5s_6 & s_1(c_{234}s_5) - c_1c_5 & s_1(c_{234}a_4 + c_{23}a_3 + c_2a_2) \\ s_{234}c_5c_6 + c_{234}s_6 & -s_{234}c_5c_6 + c_{234}c_6 & s_{234}s_5 & s_{234}a_4 + s_{23}a_3 + s_2a_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$^{R}T_H = \begin{bmatrix} n_x & o_x & a_x & p_x \\ n_y & o_y & a_y & p_y \\ n_z & o_z & a_z & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -0.707 & 0.707 & 0 & -5 \\ 0 & 0 & -1 & 0 \\ -0.707 & -0.707 & 0 & 30 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\Delta = \begin{bmatrix} 0 & -\delta z & \delta y & dx \\ \delta z & 0 & -\delta x & dy \\ -\delta y & \delta x & 0 & dz \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -0.1 & -2 \\ 0 & 0.1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad [dT] = [\Delta][T] = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0.707 & 0.707 & 0 & -5 \\ 0 & 0 & -0.1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\therefore \quad \frac{d\theta_1}{dt} = \frac{-dp_x s_1 + dp_y c_1}{(p_x c_1 + p_y s_1)} = \frac{-1(0) - 5(1)}{-5(1) + 0(0)} = 1 \ \text{(rad/sec)}$$