EEN582: Artificial Neural Networks

Optimization Problems with Solutions

References

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Problems with Solutions

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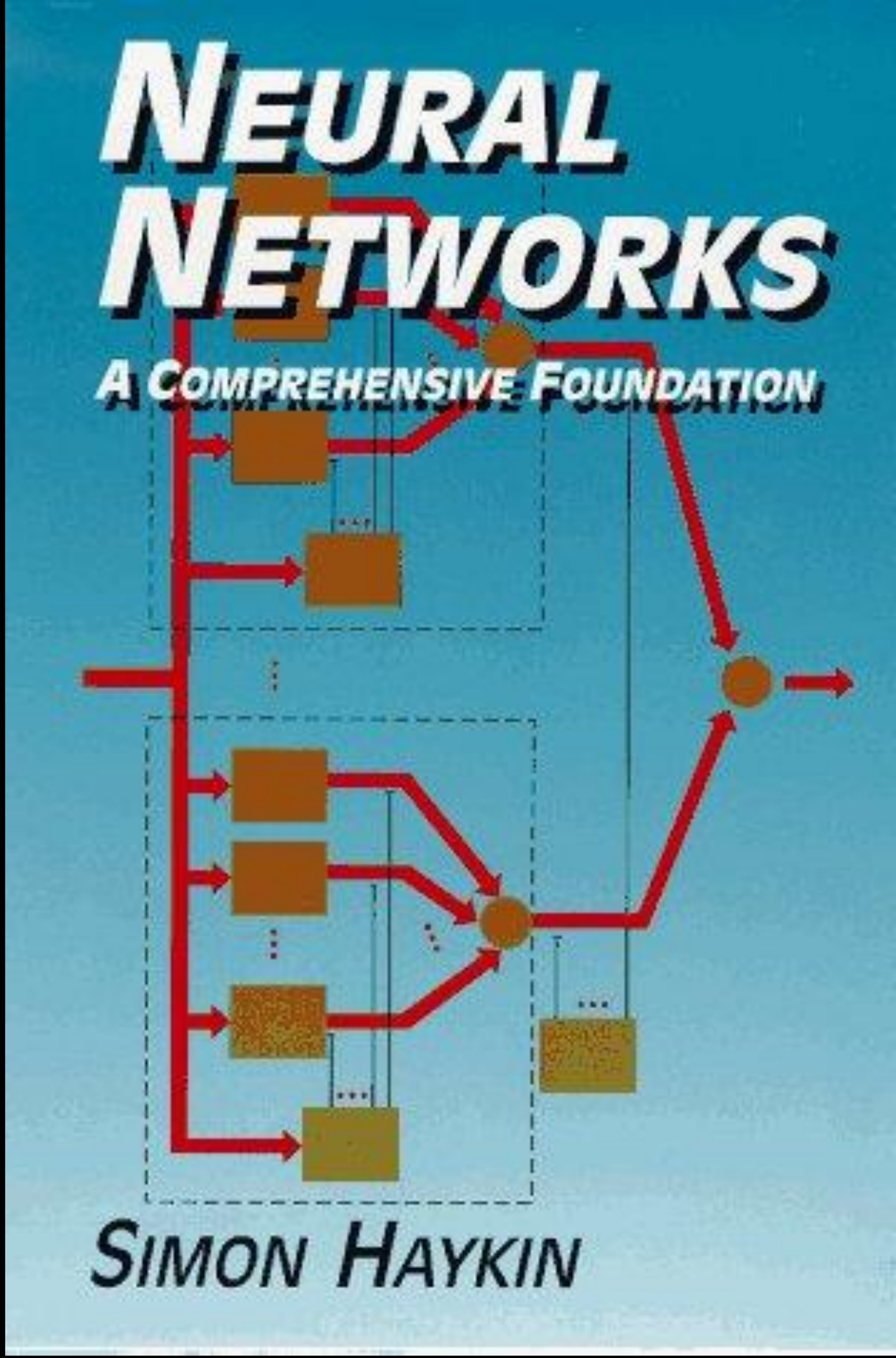


Optimization Basics
Mathematical Background for Optimization Problems

- A (mathematical) optimization problem, or just optimization problem, has the form

$$\begin{aligned} \text{minimize} \quad & f_0(\mathbf{x}) \\ \text{subject to} \quad & f_i(\mathbf{x}) \le b_i, \;\; i = 1, 2, \dots, m. \end{aligned} \tag{1}$$

- Here, the vector $\mathbf{x} = (x_1, x_2, \dots, x_n)$ is the ***optimization variable*** of the problem, the function $f_0 : \mathbb{R}^n \to \mathbb{R}$ is the ***objective function***, the functions $f_i : \mathbb{R}^n \to \mathbb{R}, i = 1, 2, \dots, m$, are the (inequality) ***constraint functions***, and the constants $b_1, b_2, \dots, b_m$ are the ***limits***, or ***bounds***, for the constraints.
- A vector $\mathbf{x}^*$ is called *optimal*, or a *solution* of the problem (1), if it has the smallest objective value among all vectors that satisfy the constraints: for any z with $f_1(\mathbf{z}) \le b_1, \dots, f_m(\mathbf{z}) \le b_m$, we have $f_0(\mathbf{z}) \ge f_0(\mathbf{x}^*)$.

Source: [3] Stephen Boyd, Lieven Vandenberghe. (2004). Convex Optimization. Cambridge University Press, Cambridge 2004

Optimization Basics

Mathematical Background for Optimization Problems

- We generally consider families or classes of optimization problems, characterized by particular forms of the objective and constraint functions.
- As an important example, the optimization problem (1) is called a *linear program* if objective and constraint functions $f_0, \ldots, f_m$ are linear, i.e., satisfy

$$f_i(\alpha\mathbf{x} + \beta\mathbf{y}) = \alpha f_i(\mathbf{x}) + \beta f_i(\mathbf{y}) \tag{2}$$

- A convex optimization problem, however, is one in which the objective and constraint functions are convex, which means they satisfy the inequality:

$$f_i(\alpha\mathbf{x} + \beta\mathbf{y}) \leq \alpha f_i(\mathbf{x}) + \beta f_i(\mathbf{y}) \tag{3}$$

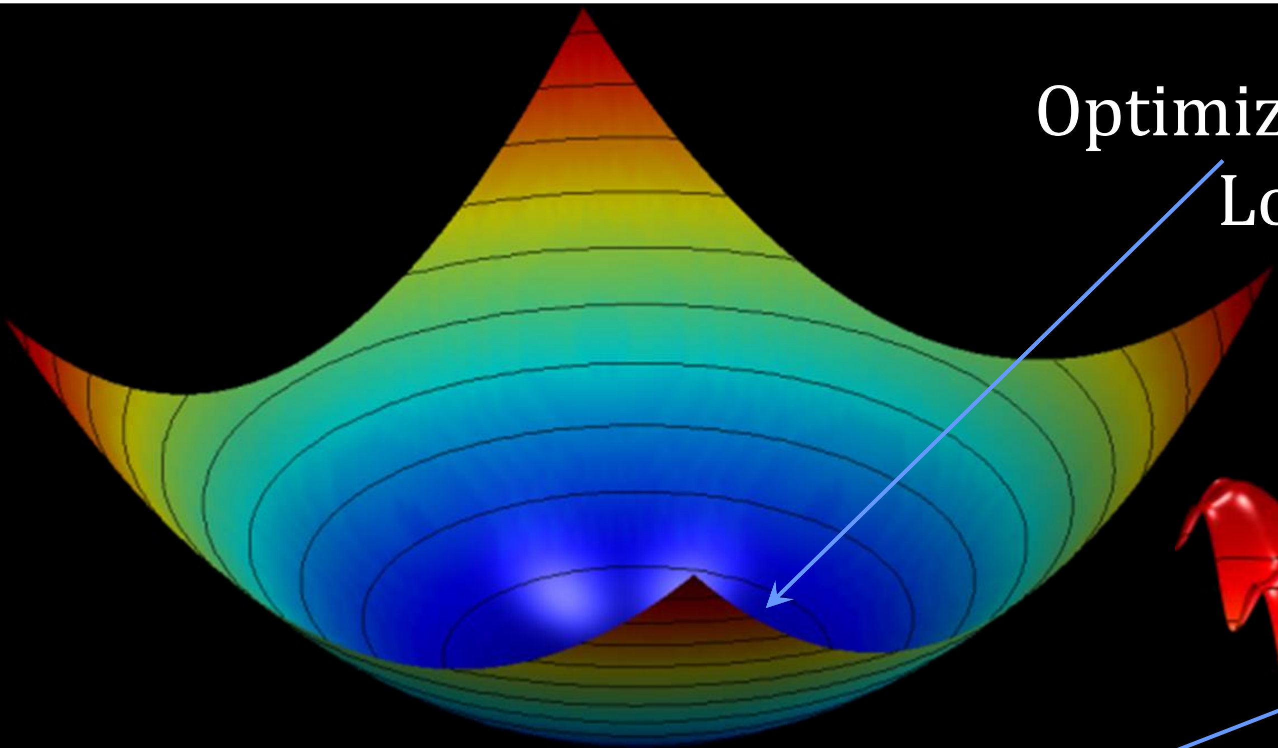
For all $x, y \in \mathbb{R}^n$ and $\alpha, \beta \in \mathbb{R}$.

- If the optimization problem is not linear, it is called a *nonlinear program*.

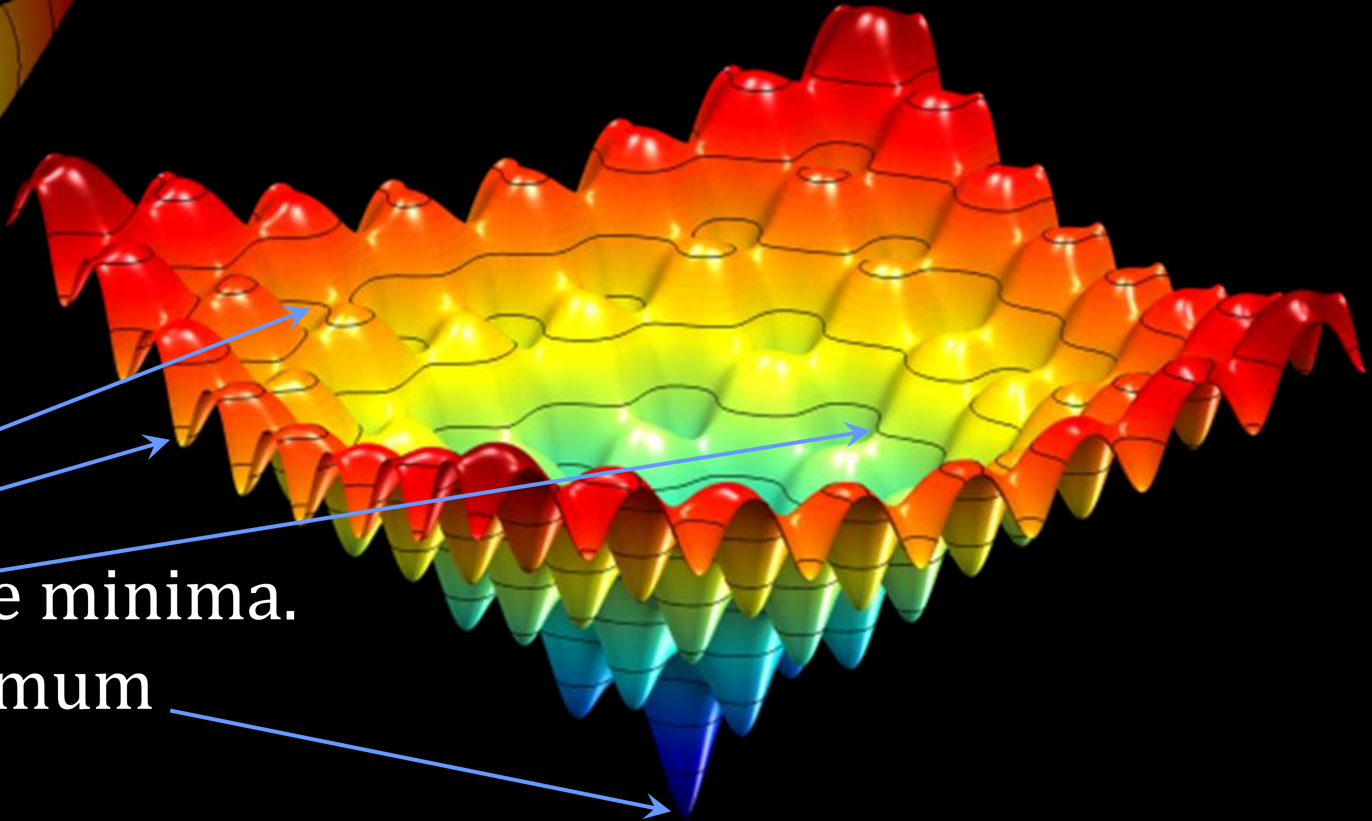
Source: [3] Stephen Boyd, Lieven Vandenberghe. (2004). Convex Optimization. Cambridge University Press, Cambridge 2004

Optimization Basics

Mathematical Background for Optimization Problems



Optimization problem with a single minimum.
Local minimum = Global minimum



Optimization problem with multiple minima.
Local minimum ≠ Global minimum

https://www.cs.bham.ac.uk/research/projects/ecb/displayDataset.php?id=2

Ex. 1: The sum of 2 positive numbers is 300 and the product is a maximum. Find these two numbers.

The first step is to write down equations describing this situation.

Let's call the two numbers x and y and we are told that the sum is 300 (this is the constraint for the problem) or,

$$x + y = 300$$

And we need to maximize the product $A = xy$. We will solve the constraint for x or y and substitute this solution into the equation of

$$y = 300 - x \Rightarrow A(x) = xy = x(300 - x) = 300x - x^2$$

We then need to find the max or min of this equation by taking the first derivative and equating to zero, that is

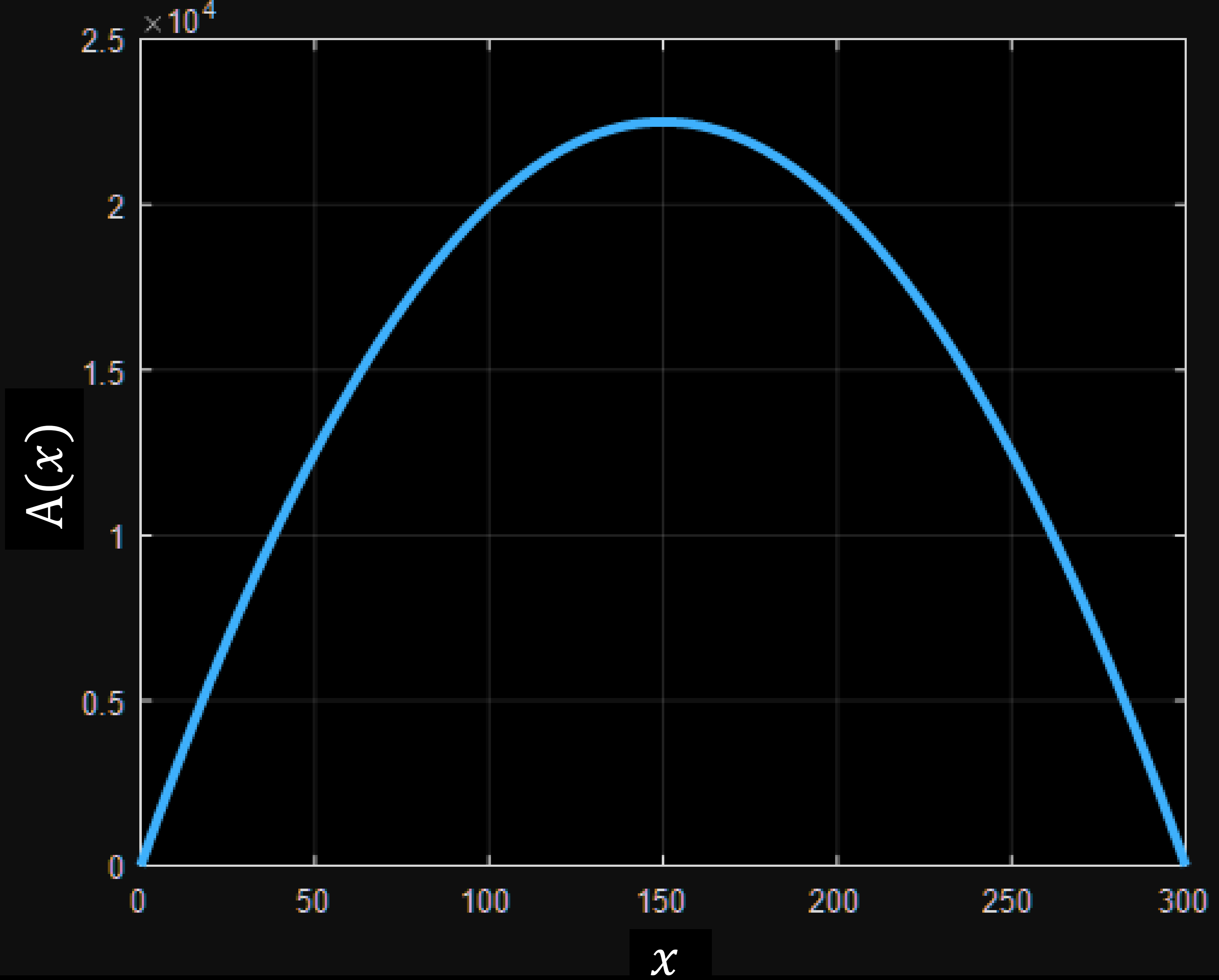
$$A'(x) = 300 - 2x = 0 \Rightarrow x = 150$$

We need to ensure that this is a maximum by taking the second derivative $A''(x) = -2$. Since the second derivative is –ve, the solution we obtained corresponds to a maximum point and maximizes the product. i.e. $X = 150$ and $y = 300 - x = 300 - 150 = 150$.

https://tutorial.math.lamar.edu/Problems/CalcI/Optimization.aspx

Graphical representation of the objective function A(x) versus optimization variable x.

The optimum (maximum) solution is at $x = 150$.



Ex. 2: Find 2 positive numbers whose product is 750 and for which sum of one and 10 times the other is a minimum.

$xy = 750$ and minimizes $S = x + 10y$.

Finding $x = 750/y$ and putting in $S(y) = 750/y + 10y$

Taking the first derivative and equating to zero,

$S'(y) = -\frac{750}{y^2} + 10 = 0 \Rightarrow y = \mp\sqrt{75} = 5\sqrt{3}$ since y is said to be positive. To check if this solution in max or min, take the second derivative of S

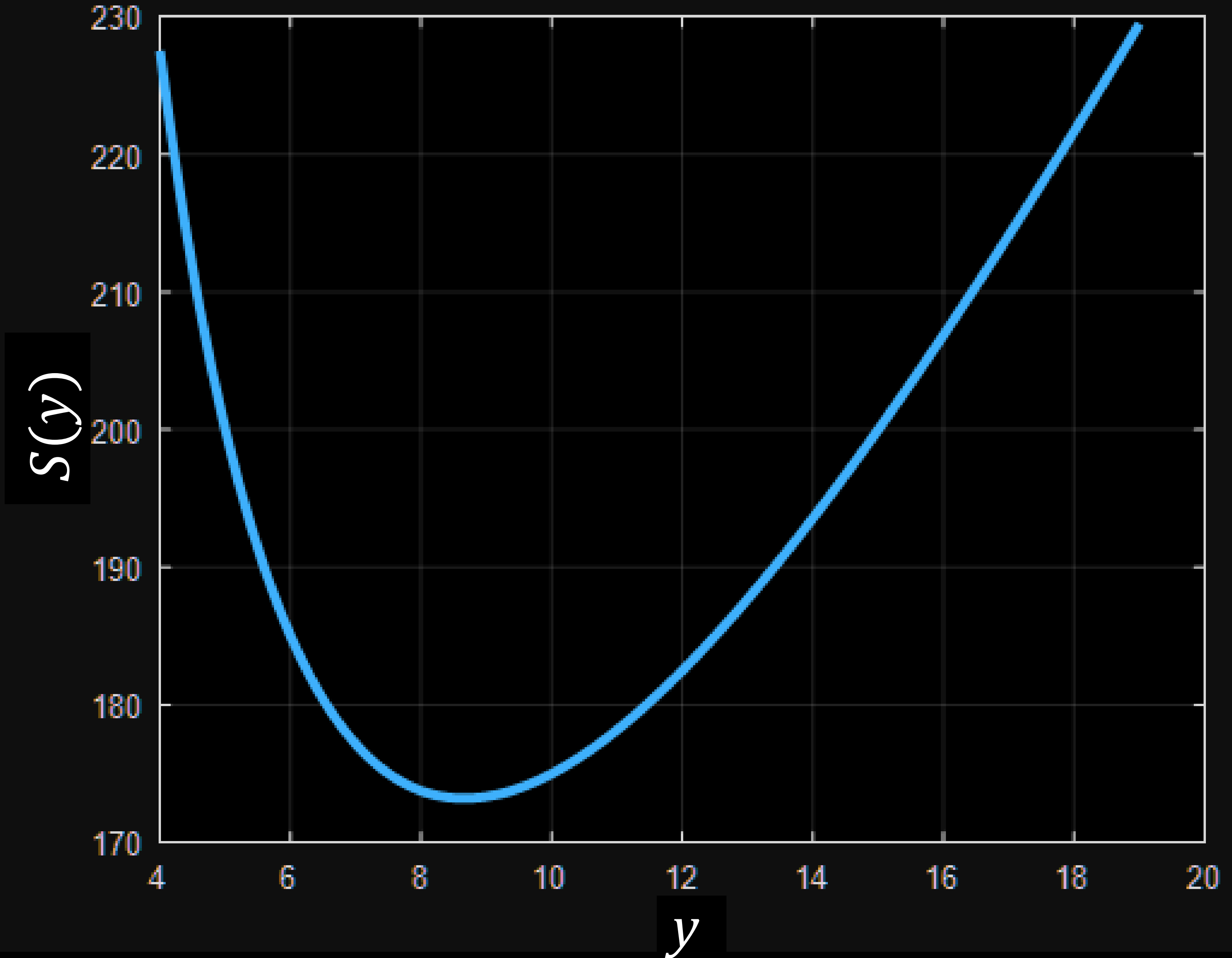
$$S''(y) = \frac{1500}{y^3}$$

Since the second derivative is >0, this is a minimum point.

The value of x corresponding to $y = 5\sqrt{3} \Rightarrow \quad x = 50\sqrt{3}$.

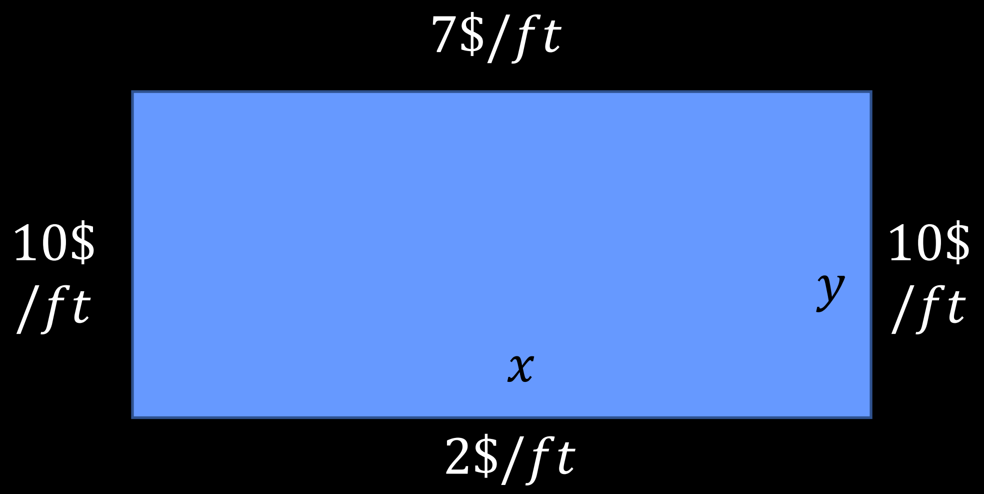
Graphical representation of the objective function $S(y)$ versus optimization variable y.

The optimum (maximum) solution is at $y = 5\sqrt{3} = 8.66$.



Ex. 3: We are going to fence in a rectangular field. If we look at the field from above the cost of the vertical sides are \$10/ft, the cost of the bottom is \$2/ft and the cost of the top is \$7/ft. If we have \$700 determine the dimensions of the field that will maximize the enclosed area.

Sketch the area mentioned in the problem and we have



Since we have \$700 to spend and so the cost of the material will be the constraint for this problem. The cost for the material will then be written as,

$$700 = 10y + 2x + 10y + 7x = 20y + 9x \Rightarrow$$

$$y = 35 - \frac{9}{20}x$$

To maximize the area $A = xy$, let's solve the constraint for y in by taking derivative of

$$A(x) = x\left(35 - \frac{9}{20}x\right) = \left(35x - \frac{9}{20}x^2\right)$$

$$A'(x) = 35 - \frac{9}{10}x = 0 \Rightarrow x = \frac{350}{9}$$

The second derivative of A

$$A''(x) = -\frac{9}{10} = 0 \Rightarrow x = \frac{350}{9}$$

Since the second derivative is always negative and so $A(x)$ will always be concave down and so the critical point obtained is a relative maximum and hence must be the value that gives a maximum.

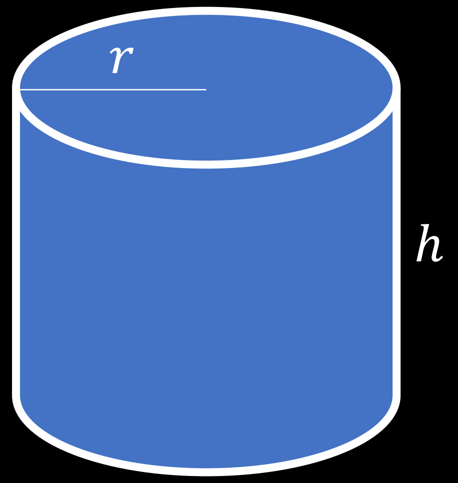
The corresponding point for y is then

$$y = 35 - \frac{9}{20}x = 35 - \frac{9}{20}\frac{350}{9} = \frac{35}{2}$$

Hence, to maximize the land to buy for \$700, wee need to make the dimensions of the square as

$$x = \frac{350}{9} \text{ and } y = \frac{35}{2}$$

Ex. 4: Construct a cylindrical can with a bottom but no top that will have a volume of 30 cm^3. Determine the can dimensions that will minimize amount of material needed to construct the can.



With a clear plot of the can. We are now ready set up the constraint and equation that we are being asked to optimize.

The first constraint is the volume of the can must be 30 cm3. We are then asked to minimize the amount of material needed to construct the volume of the can, $A = 2\pi rh + \pi r^2$, since the can will have no top and so the second term will only be for the area of the bottom of the can.

Solving for the constraint that the volume of the can should be

$$30 = \pi r^2 h \Rightarrow h = \frac{30}{\pi r^2}$$

Substituting this into the amount of material function

$$A(r) = 2\pi r\left(\frac{30}{\pi r^2}\right) + \pi r^2 = \frac{60}{r} + \pi r^2$$

https://tutorial.math.lamar.edu/Problems/CalcI/Optimization.aspx

To find the critical point, equate the first derivative to zero,

$$A'(r) = \left(\frac{-60}{r^2}\right) + 2\pi r = \frac{2\pi r^3 - 60}{r^2} = 0 \;\Rightarrow\; r = \left(\frac{60}{2\pi}\right)^{\frac{1}{3}} = 2.1216 \text{ cm}$$

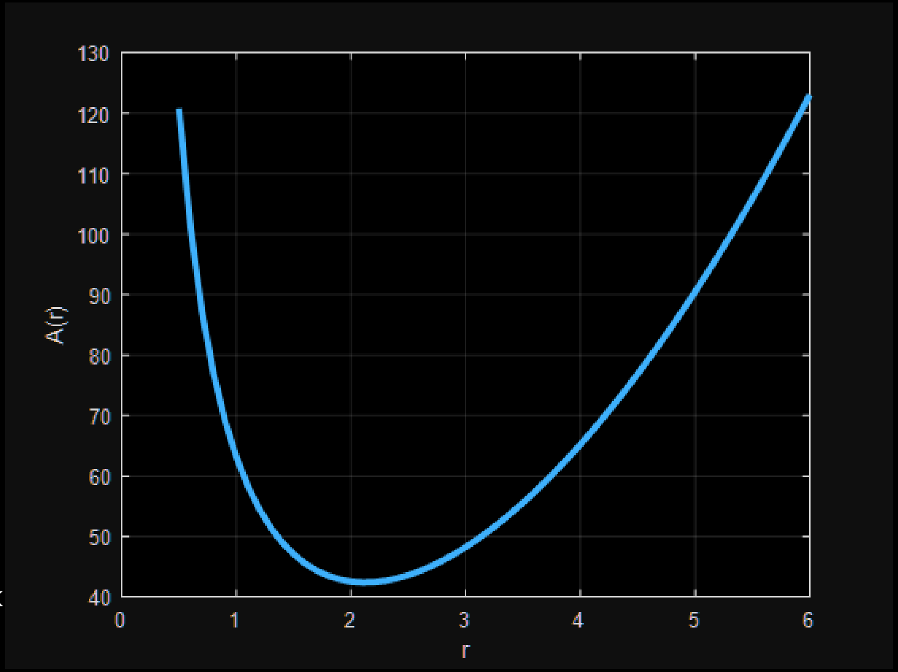
We need the second derivative to verify that this is indeed a minimum. So,

$$A''(r) = \left(\frac{120}{r^3}\right) + 2\pi$$

Since second derivative is positive, it shows tendency to increase, hence a minimum point.

Now, with $r = 2.1216$ cm in our hand, we can find the values of h that will minimize the material needed

$$h = \frac{30}{\pi(2.1216)^2} = 2.1215 \text{ cm}$$



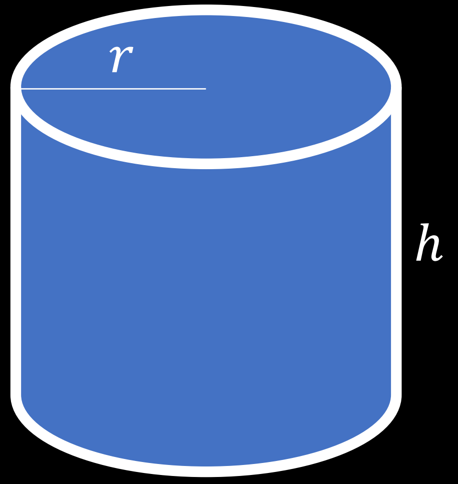
$A(r)$ versus r.

The optimum (minimum) solution is at $r = 2.1216$ cm.

https://tutorial.math.lamar.edu/Problems/CalcI/Optimization.aspx

Ex. 5: A cylindrical can, with a top lid, must contain V cm^3 of liquid. (A typical can of soda, for example, has V = 355 cm^3). What dimensions (height and radius) will minimize the cost of metal needed to construct the can?

Stage I. Develop the function



Step 1

In Optimization problems, always begin by sketching the situation. The problem asks us to minimize the cost of the metal used to construct the can, so we've shown each piece of metal separately:

- the can's circular top,
- cylindrical side, and
- circular bottom.

We've labeled the can's height *h* and its radius *r*. We're looking for the values of *h* and *r* (in terms of the volume *V*) that will minimize the cost of constructing the can.

Step 2

Having drawn the picture, now write an equation for the quantity to be optimized.

In this problem, for instance, we want to minimize the cost of constructing the can, which means we want to use as little metal as possible.

Hence, we want to minimize the can's surface area. So, let's write an equation for that total surface area:

$$A_{total} = A_{top} + A_{cylinder} + A_{bottom} = \pi r^2 + 2\pi rh + \pi r^2 = 2\pi r^2 + 2\pi rh$$

Now we have written the equation for the quantity to be minimized i.t.o. the relevant parameters (r and h).

Step 3

To solve optimization problems: we have to use detailed information given in the problem and rewrite the equation developed in Step 2 to be in terms of a single variable.

Our equation for A_{total}, has two variables, r and h. We must eliminate one of them in order to proceed. We arbitrarily decide to eliminate h and keep r and hence need to describe h in terms of r so we can substitute for h as a variable.

From the equation describing volume of cylinder V i.t.o. r and h, ve get

$$V = \pi r^2 h \Rightarrow h = \frac{V}{\pi r^2}$$

Putting h in A_{total}, we get

$$A_{total}(r) = 2\pi r^2 + 2\pi r h = 2\pi r^2 + 2\pi r\left(\frac{V}{\pi r^2}\right) = 2\pi r^2 + \left(\frac{2V}{r}\right)$$

This equation emphasize that $A(r)$ is a function of only the single variable r, and we've dropped the subscript A_{total} since we no longer need it.

Stage II: Maximize or Minimize the Function

An optimization problem is merely a max/min problem where we first develop the function we're going to maximize or minimize.

The remaining steps are exactly the same as they are for the max/min problems.

Step 4

We want to minimize the following function by taking the derivative $A'(r)$ and then finding the critical point at $A'(r) = 0$,

$$A'(r) = 4\pi r - \frac{2V}{r^2} = 0$$

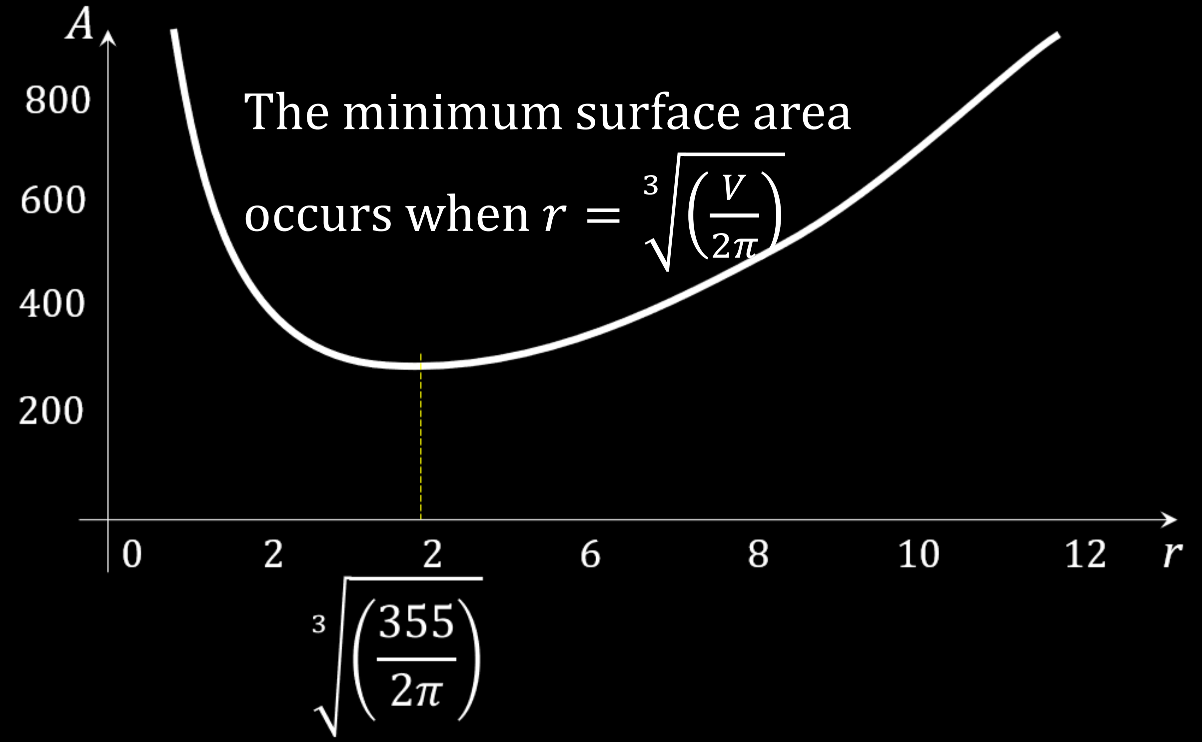
Solving, we get, $r^3 = \frac{V}{2\pi} \Rightarrow \ r = \sqrt[3]{\left(\frac{V}{2\pi}\right)}$

Step 5.

Next, we must verify that this critical point represents a minimum for the surface area by using the Second Derivative Test.

$$A''(r) = 4\pi + \frac{4V}{r^3}$$

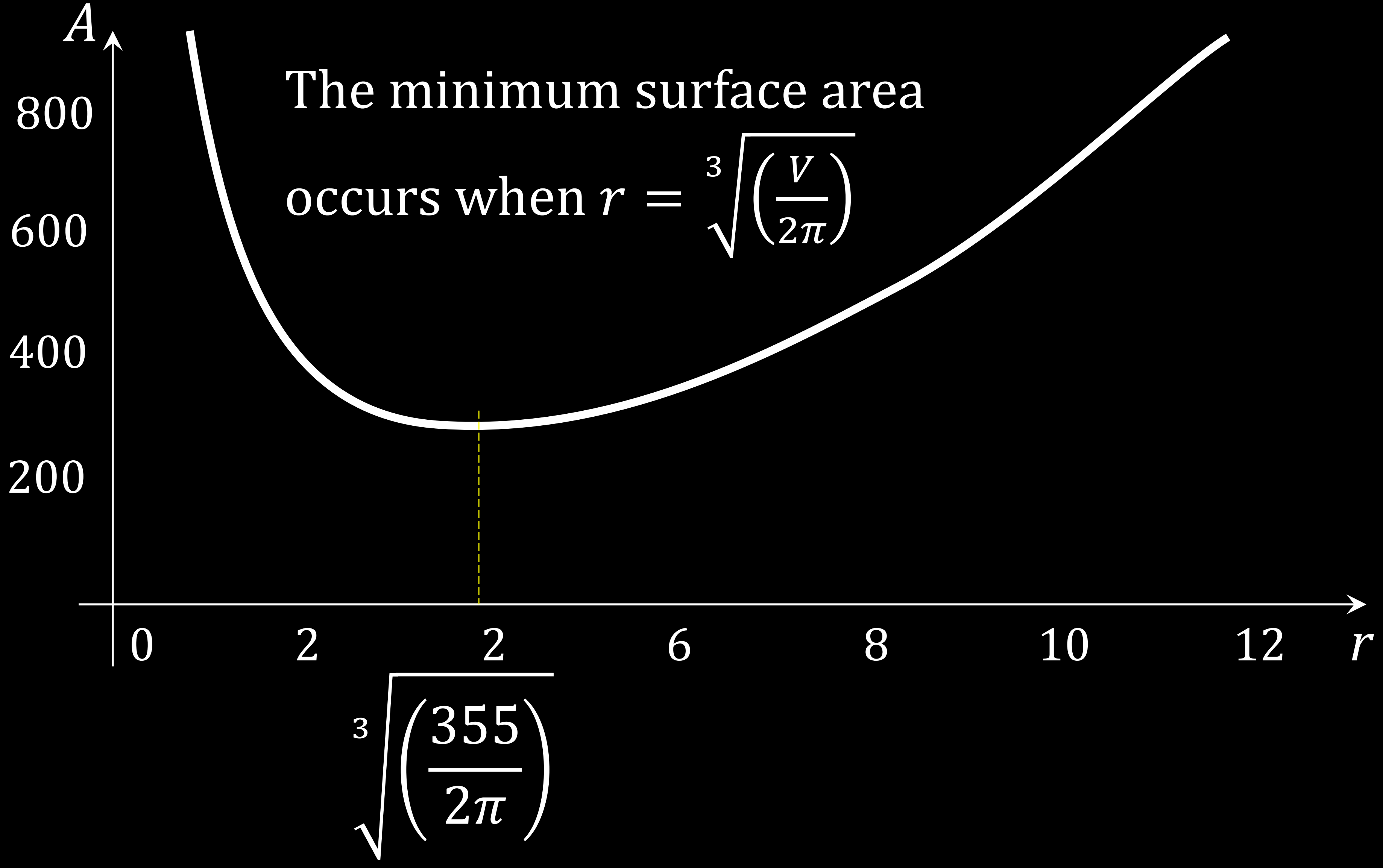
Since , $r > 0$, this second derivative is always positive. That is, the graph of $A(r)$ versus r is always concave up. Hence this single critical point gives us a minimum.



Step 6

As we found the critical point corresponding to the can's minimum surface area (and hence minimized the cost), we can finish answering the question by finding the dimensions (r and h) of the least-expensive can using

$$r = \sqrt[3]{\left(\frac{V}{2\pi}\right)} \text{ in } h = \frac{V}{\pi r^2} = \frac{V}{\pi\left(\sqrt[3]{\left(\frac{V}{2\pi}\right)}\right)^2} = \left(2^{\frac{2}{3}}\right)\sqrt[3]{\left(\frac{V}{\pi}\right)} = 2r$$


A
800
600
400
200
The minimum surface area
occurs when $r = \sqrt[3]{\left(\frac{V}{2\pi}\right)}$
0
2
2
6
8
10
12
r
$\sqrt[3]{\left(\frac{355}{2\pi}\right)}$

Optimization Problems in Economics

Example 6

A game console manufacturer determines that in order to sell x units, the price per one unit (in dollars) must decrease by the linear law (the demand function)

$$p(x) = 500 - 0.1x \text{ (\$/device)}$$

The manufacturer also determines that the cost depends on the volume of production and includes a fixed part \$100,000 and a variable part $100x$, that is

$$C(x) = 100000 + 100x$$

What price per unit must be charged to get the maximum profit?

Optimization Problems in Economics

Solution 6

The total revenue is given by $R(x) = xp(x) = x(500 - 0.1x) = 500x - 0.1x^2$

The profit is determined by the formula

$$P(x) = R(x) - C(x) = 500x - 0.1x^2 - 100000 - 100x$$
$$= 400x - 0.1x^2 - 100000$$

Finding the derivative $P'(x) = 400 - 0.2x$ and the critical point at $P'(x) = 0$, results in $x = 2000$.

We use the Second Derivative $P''(x)$ to justify that the critical point is a max.

$P''(x) = -0.2$, which is <0, hence the price is maximized when $x = 2000$ consoles are sold. The price per unit will then be $p(x = 2000) = 500 - 0.1 \times 2000 = \$300/\text{device}$.

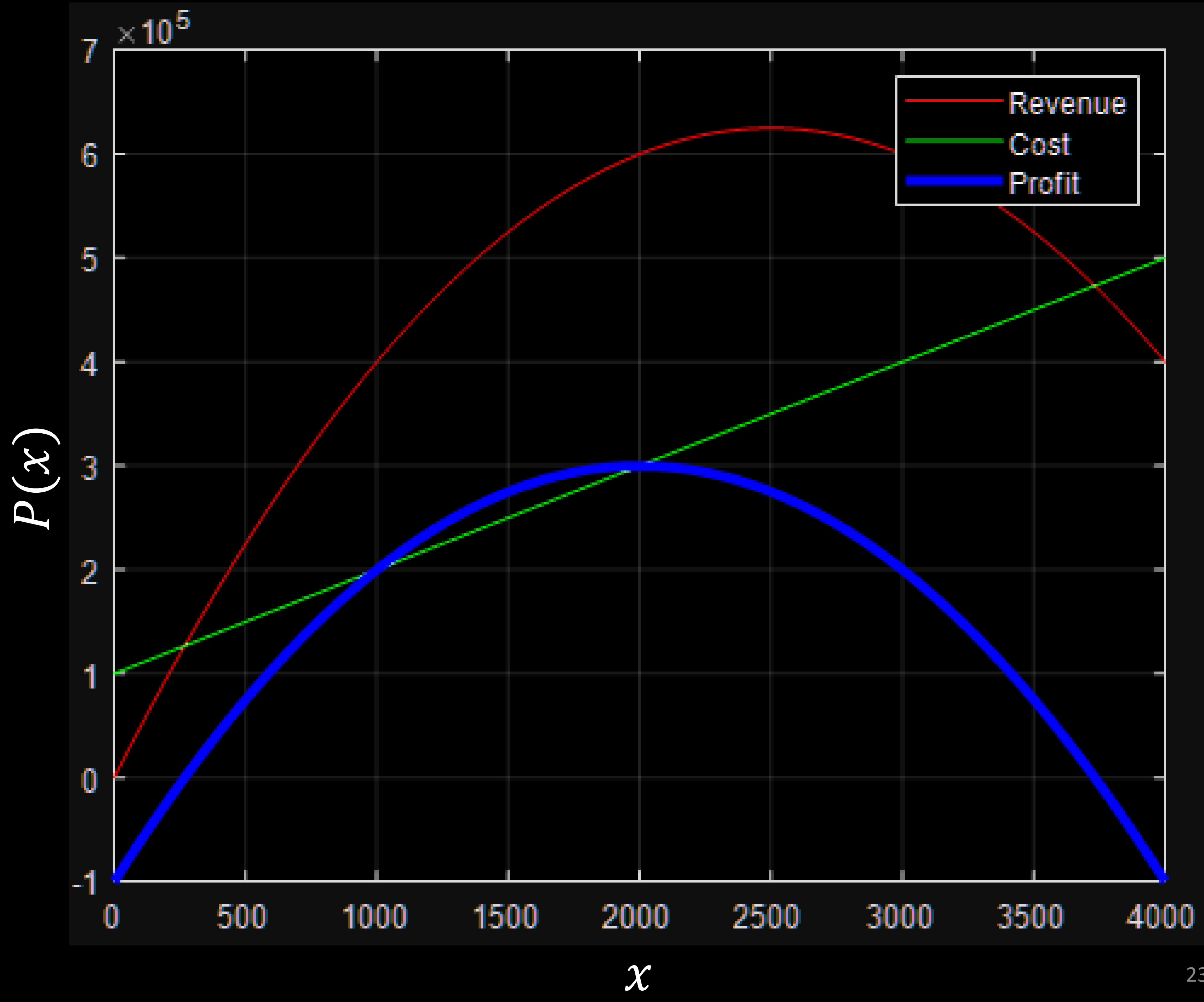
Optimization Problems in Economics

```
% OPTIMIZATION PROBLEM IN ECONOMICS
clear;
x = 0:0.1:4000;
p = 500 - 0.1*x;
R = 500*x - 0.1*x.^2;   % The total revenue
Cx = 100000 + 100*x;
Px = 400*x - 0.1*x.^2 - 100000; % The profit
hold off;
plot(x,R,'r','linewidth',1);
hold on;
plot(x,Cx,'g','linewidth',1);
plot(x,Px,'b','linewidth',3);
xlabel('x'); ylabel('P(x)'); grid;
legend('Revenue', 'Cost', 'Profit');
figure(1)
```

https://www.math24.net/optimization-problems-economics#example1

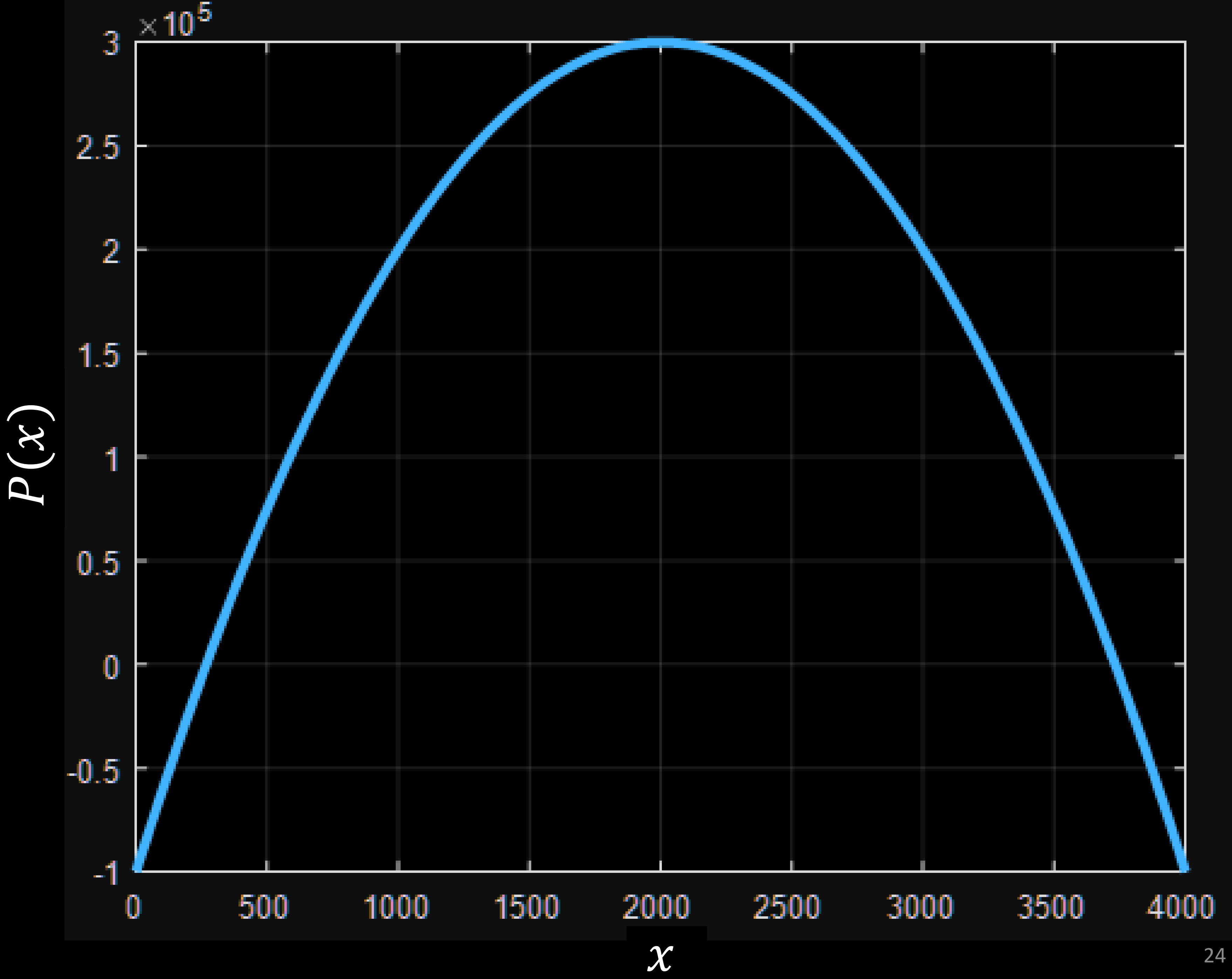
Graphical representation of the total profit (the objective function), $P(x)$ versus the number of consoles (optimization variable) x.

The optimum (maximum) solution is at $x = 2000$.



Graphical representation of the total profit (the objective function), $P(x)$ versus the number of consoles (optimization variable) x.

The optimum (maximum) solution is at $x = 2000$.



Optimization Problems in Healthcare

Example 7: How much daily exercise is optimal? Suppose every human heart has a fixed number of beats before it stops working. How should one optimally use these heart beats to have as long a life as possible?

Staying in bed reduces heart beats and exercise increases it. However, a trained heart could beat much slower. So, is there an optimal value?

Solution 7: Suppose that a trained heart beats 120 times a minute during exercise and 80 times a minute when the person is at rest. If the person exercise a fraction of time (say x), the persons heart beats per minute will will be represented by

$$f(x) = 120x + g(x)(1 - x)$$

where the unknown function in $g(x)$ should be close to 80 for small x, when the person is at rest and around 50 for x close to 1 when the person is well trained .

Optimization Problems in Healthcare

Since exercising every day decreases the heart bets at rest, a simple model for $g(x)$ would be exponential decay as shown by

$$g(x) = 50 + 30e^{-100x}$$

where the unknown function in $g(x)$ should be close to 80 for small x.

Combining $f(x)$ and $g(x)$, we get

$$f(x) = 120x + (50 + 30e^{-100x})(1 - x)$$

$$= 120x + 50 + 30e^{-100x} - 50x - 30xe^{-100x}$$

Taking the first derivative

$$f'(x) = 120 - 30 \times 100 - 50 - 30e^{-100x} - 30 \times 100xe^{-100x} = 0$$

https://www.unige.ch/~gander/teaching/polycopie.pdf

Optimization Problems in Healthcare

$$f(x) = 120x + 50 + 30e^{-100x} - 50x - 30xe^{-100x}$$

$$f'(x) = 70 - 3000e^{-100x} - 30e^{-100x} - 3000xe^{-100x} = 0$$

$$3000e^{-100x} + 30e^{-100x} + 3000xe^{-100x} = 70$$

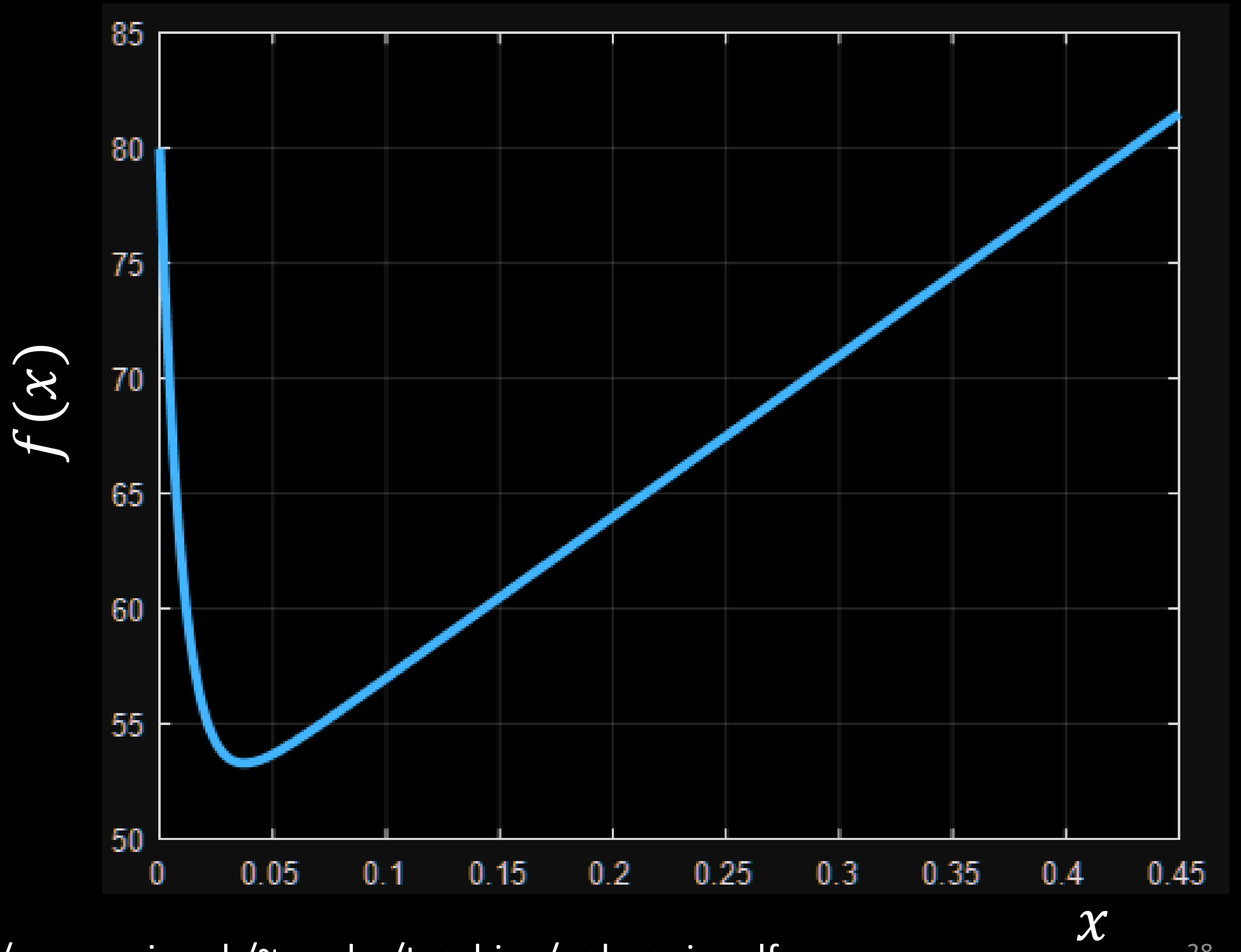
$$(3030 + 3000x)e^{-100x} = 70$$

The solution for this kind of equations is normally very difficult but the value of x that will give the minimum value of $f(x)$ can be found from the graph of $f(x)$ nersus x shown in the next page.

Optimization Problems in Healthcare

Figure: The number of heart beats per minute $f(x)$, versus percentage of time x, the person exercises every day.

The minimum value of $f(x) = 53.3$ is reached at $x = 0.037$

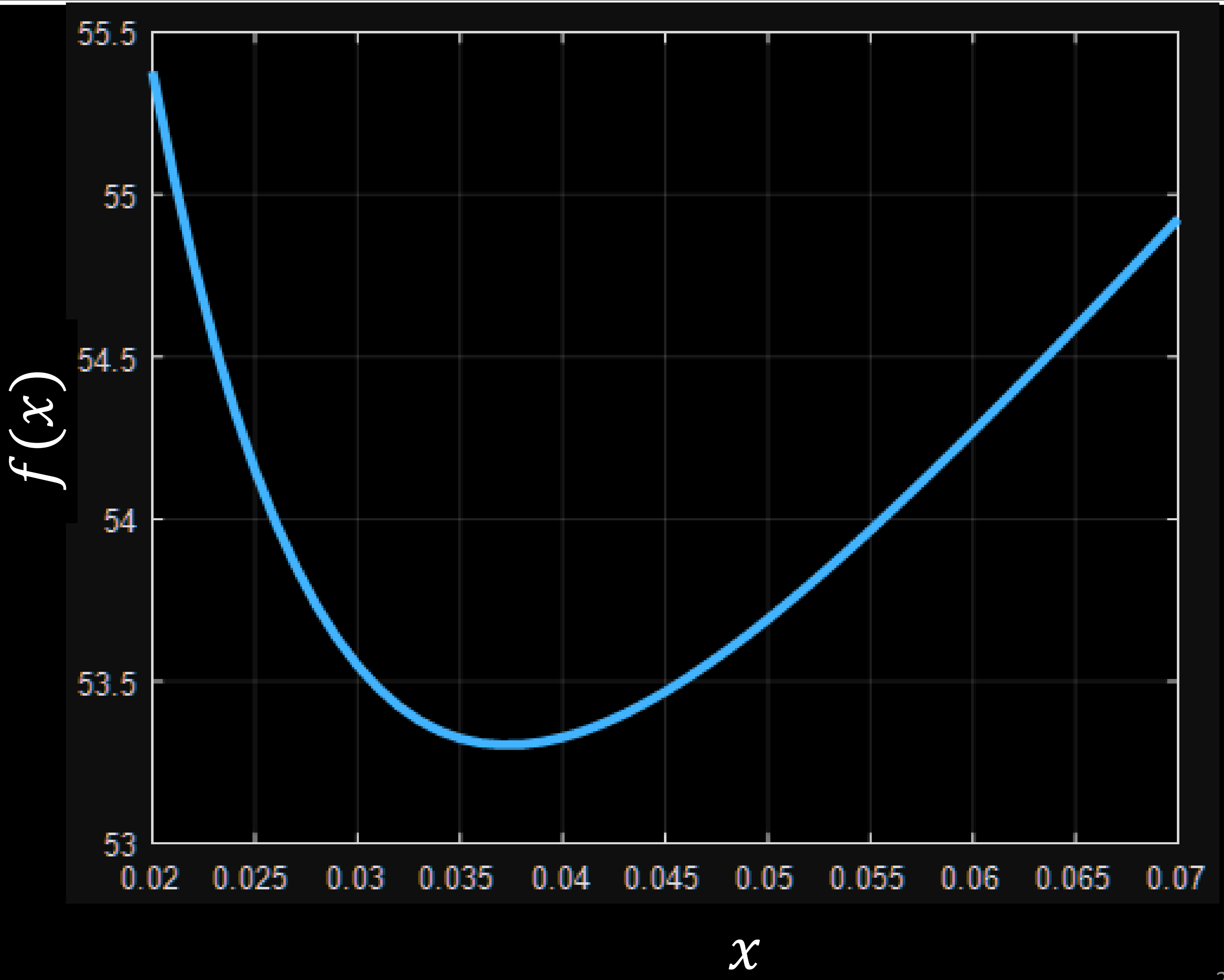


https://www.unige.ch/~gander/teaching/polycopie.pdf

Optimization Problems in Healthcare

```
% OPTIMIZATION PROBLEM PLOT
clear;
x = 0.02:0.001:0.07;
gx = 50 + 30*exp(-100*x);
fx = 120*x + gx .* (1-x);
plot(x,fx,'linewidth',3);
xlabel('x'); ylabel('f(x)'); grid;
figure(1)
```

The figüre represents the magnified version of the plot with a focus on the minimum value.



Example 8: Work Rate for Breathing

The work rate of breathing is the sum of resistive (nonelastic, W_{ne}) work and elastic (compliance, W_e) work.

$$W(f) = W_{ne}(f) + W_e(f) = af + b/f$$

The sum reaches a minimum at some particular frequency, which can be calculated by taking the first derivative and equating to zero as follows:

$$W'(f) = \frac{d}{df}(af + bf^{-1}) = a - bf^{-2} = 0 \Rightarrow f^{-2} = \frac{a}{b} \Rightarrow f^2 = \frac{b}{a}, f = \sqrt{\frac{b}{a}}$$

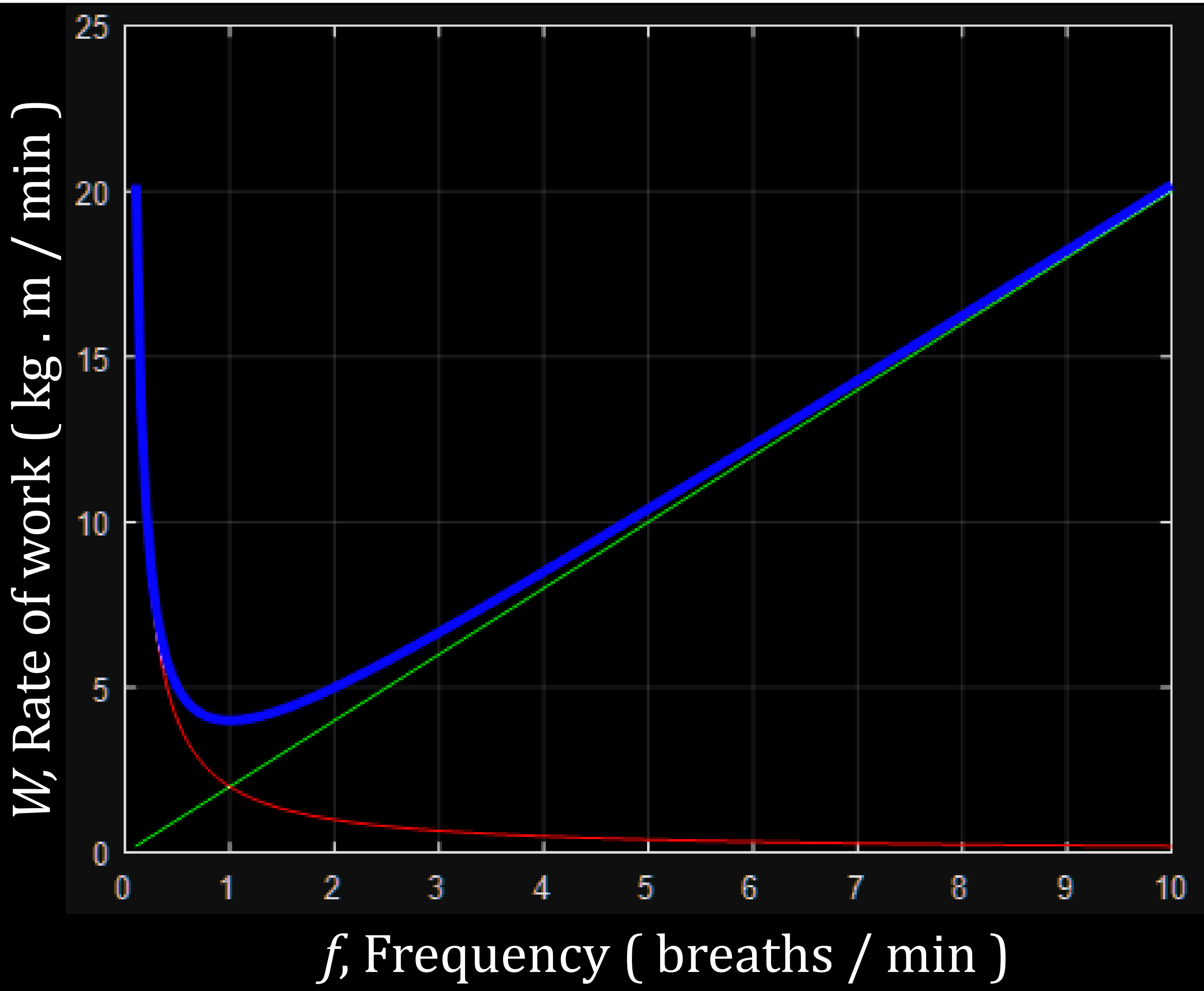
If we arbitrarily chose $a = b = 2$, then $f = 1$ is the solution since frequency of breathing could not be negative.

Since $W''(f) = 2bf^{-3}$ is always positive, indeed the above solution shows a minimum value as shown in the next figure

https://journals.physiology.org/doi/pdf/10.1152/jappl.1950.2.11.592

Example 8: Work Rate for Breathing

```
% WORK RATE OF BREATHING
clear;
a = 2;
b = 2;
s = 0.1:0.05:10;
Ene = a*s;      % non-elastic work rate
Ee = b ./ s;    % elastic work rate
E = Ene + Ee;   % total energy is sum of eleastic
                % and nonelastic work rates
hold off;
plot(s,Ene,'g'); hold on; plot(s,Ee,'r')
plot(s,E,'b','linewidth',3);
xlabel('speed'); ylabel('Work Rate'); grid;
figure(1)
```



https://journals.physiology.org/doi/pdf/10.1152/jappl.1950.2.11.592

More Optimization Problems in Mathematics

https://www.youtube.com/watch?v=lx8RcYcYVuU

Example 9: Maximum of Product of x and y

Find the two numbers whose sum is equal to 60 and whose product is maximum.

$$s = x + y \quad \text{(Constraint)}$$

$$p = x * y \quad \text{(objective)}$$

$$s = x + y = 60 \quad \Rightarrow \quad y = 60 - x$$

$$p = x * y = x * (60 - x) = 60x - x^2$$

$$p' = 60 - 2x = 0 \quad \Rightarrow \quad x = 30$$

$$s = x + y \quad \Rightarrow \quad 60 = 30 + y \quad \Rightarrow \quad y = 60 - 30 = 30$$

Hence the optimal values $x = 30$ and $y = 30$.

Example 10: …

…

…

…