EENG226
Signals and Systems

Chapter 2
Time-Domain Representations of Linear Time-Invariant Systems

Interconnection of LTI Systems

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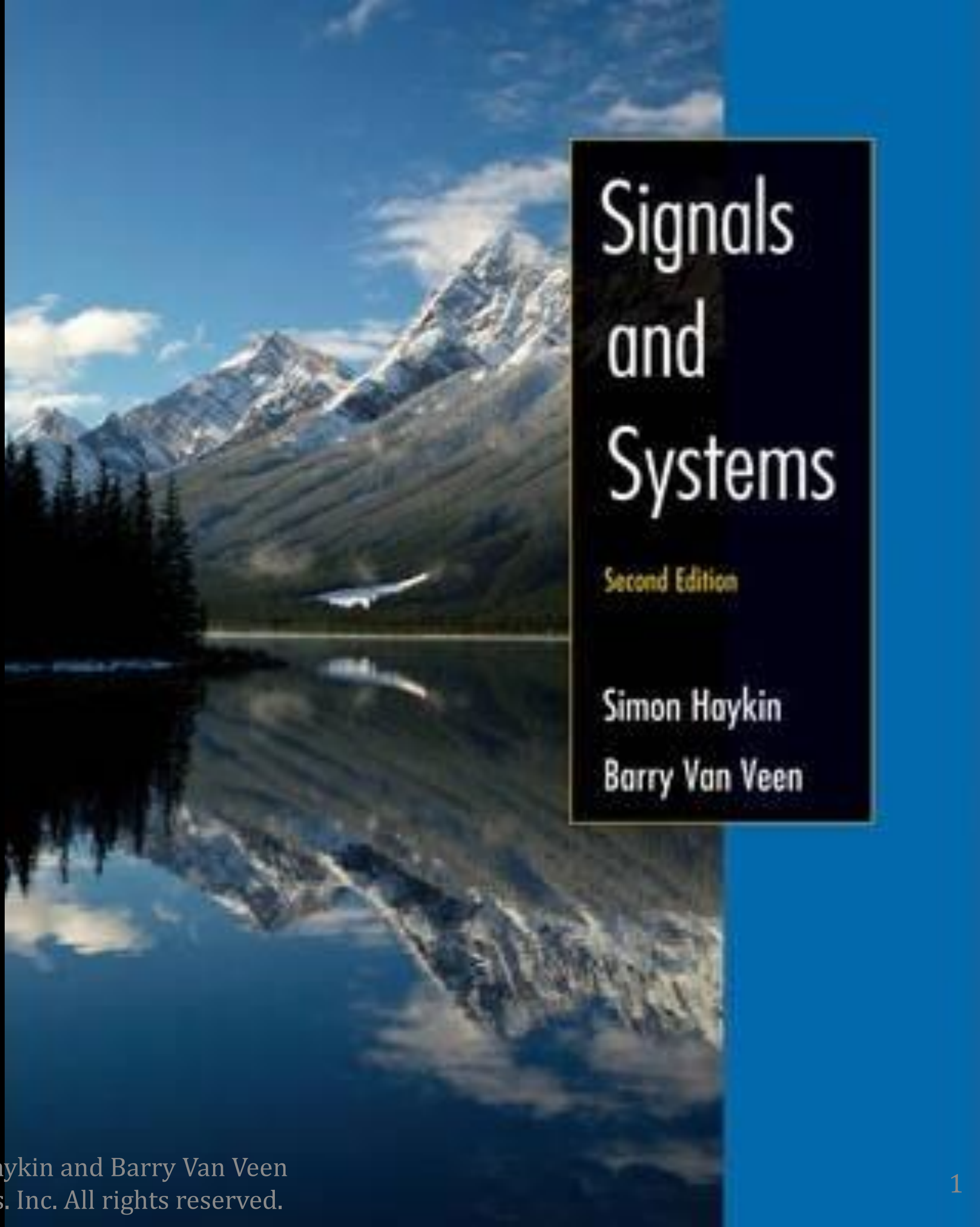

Chapter 2

Time-Domain Representations of Linear Time-Invariant Systems

Objectives of this chapter

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2.6 Interconnections of LTl Systems

2.6.1 Parallel Connection of LTI Systems

- Consider two LTI systems with impulse responses $h_1(t)$ and $h_2(t)$ connected in parallel, as illustrated in Fig. 2.18(a), the output of this connection of systems, $y(t)$, is the sum of the outputs of the two systems:

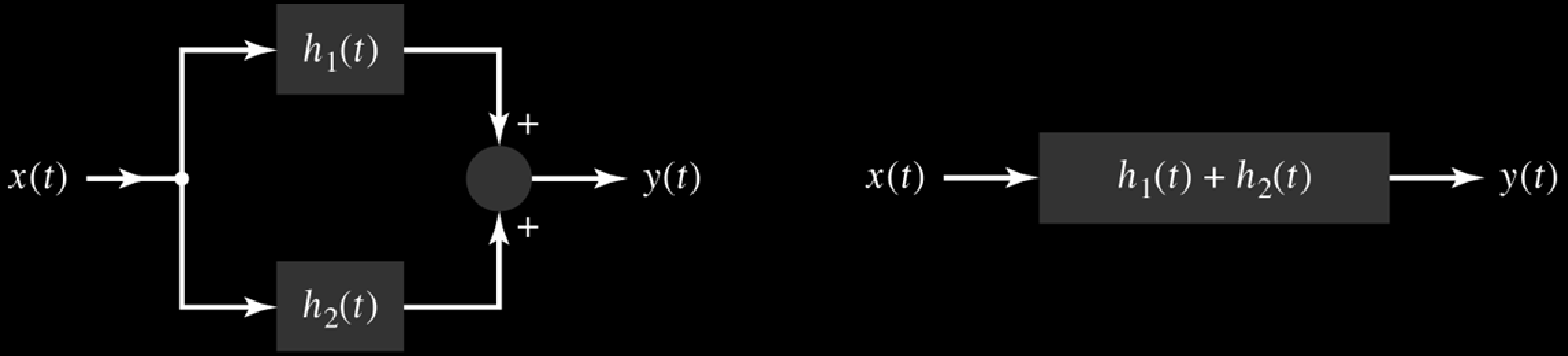
$$y(t) = y_1(t) + y_2(t) = x(t) * h_1(t) + x(t) * h_2(t) =$$

- Substituting, we get

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h_1(\mathrm{t}-\tau)d\tau + \int_{-\infty}^{\infty} x(\tau)h_2(\mathrm{t}-\tau)d\tau = \int_{-\infty}^{\infty} x(\tau)h(\mathrm{t}-\tau)d\tau$$

- Where $h\,(t\text{-}\tau) = h_1(t\text{-}\tau) + h_2(t\text{-}\tau)$, is the impulse response of the equivalent system.

Figure 2.18 (p. 128)
Interconnection of two LTI systems.
(a) Parallel connection of two systems.
(b) Equivalent system.



(a) (b)

- Identical results hold for the discrete-time case:

$$x[n] * h_1 + x[n] * h_2 = x[n] * \{h_1[n] + h_2[n]\} \qquad (2.16)$$

2.6.2 Cascade Connection of Systems

- Consider the cascade connection of two LTI systems, as illustrated in Fig. 2.19(a). Let $z(t)$ be the output of the first system and therefore the input to the second system in the cascade. The output is expressed in terms of $z(t)$ as

$$y(t) = z(t) * h_2(t) \tag{2.17}$$

- Substituting for $z(t)$, we get

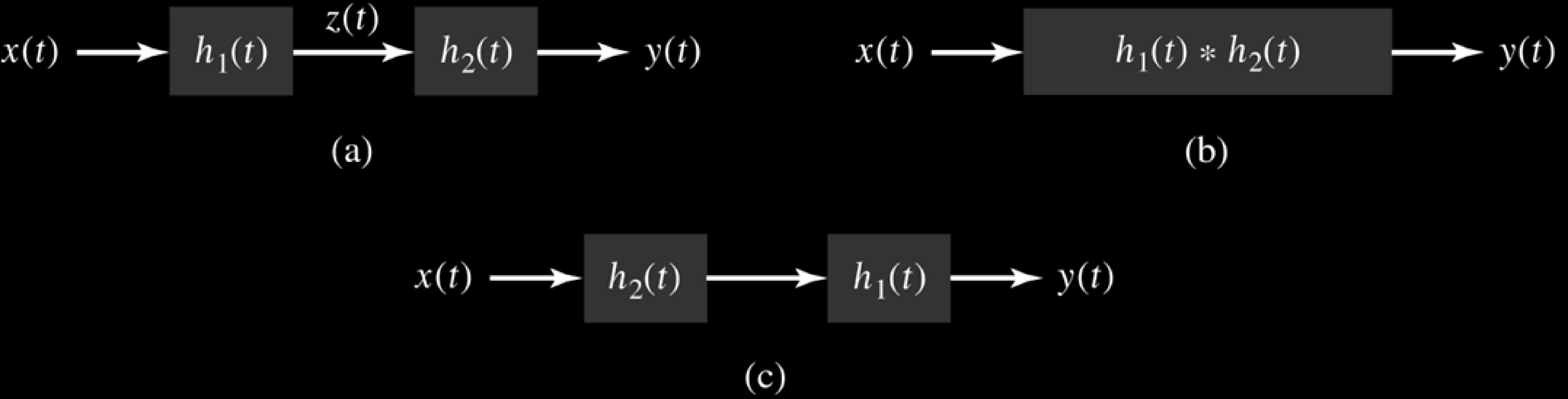
$$y(t) = \int_{-\infty}^{\infty} z(\tau)h_2(t-\tau)d\tau$$

- Putting for $z(t)$, we get

$$y(t) = \iint_{-\infty}^{\infty} x(v)h_1(\tau - v)h_2(t-\tau)dvd\tau = \int_{-\infty}^{\infty} x(v)h(t-\tau)dv = h(t) * h(t)$$

*see Fig. 2.19.b

Figure 2.19 (p. 128)
Interconnection of two LTI systems. (a) Cascade connection of two systems. (b) Equivalent system. (c) Equivalent system: Interchange system order.



EXAMPLE 2.11 EQUIVALENT SYSTEM TO FOUR INTERCONNECTED SYSTEMS Consider the interconnection of four LTI systems, as depicted in Fig. 2.20. The impulse responses of the systems are

$$h_1[n] = u[n],$$
$$h_2[n] = u[n+2] - u[n],$$
$$h_3[n] = \delta[n-2],$$

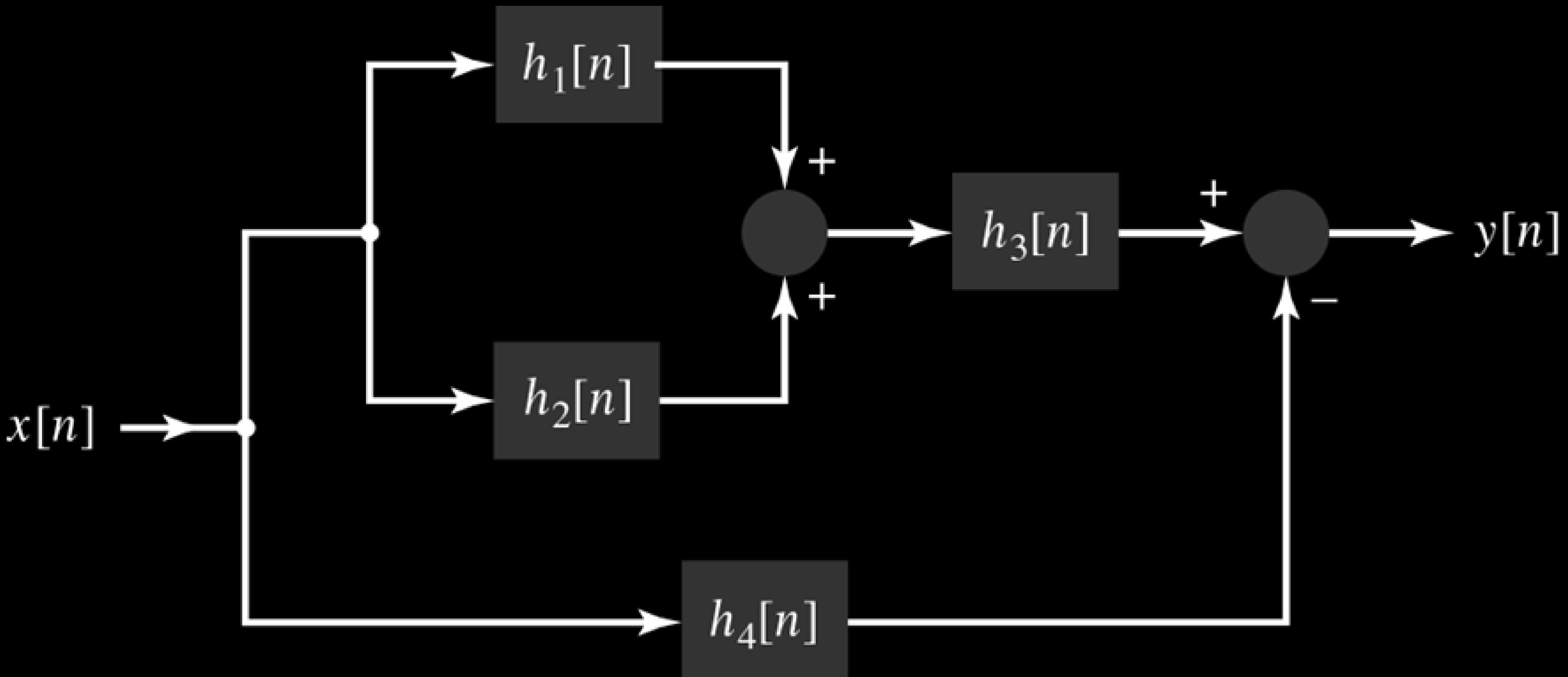
and

$$h_4[n] = \alpha^n u[n].$$

Find the impulse response $h[n]$ of the overall system.

Solution: We first derive an expression for the overall impulse response in terms of the impulse response of each system. We begin with the parallel combination of $h_1[n]$ and $h_2[n]$. The distributive property implies that the equivalent system has the impulse response $h_{12}[n] = h_1[n] + h_2[n]$, as illustrated in Fig. 2.21(a). This system is in series with $h_3[n]$, so the associative property implies that the equivalent system for the upper branch has the impulse response $h_{123}[n] = h_{12}[n] * h_3[n]$. Substituting for $h_{12}[n]$ in this expression, we have $h_{123}[n] = (h_1[n] + h_2[n]) * h_3[n]$, as depicted in Fig. 2.21(b). Last, the upper branch

Figure 2.20 (p. 131)
Interconnection of systems for Example 2.11.



is in parallel with the lower branch, characterized by $h_4[n]$; hence, application of the distributive property gives the overall system impulse response as $h[n] = h_{123}[n] - h_4[n]$. Substituting for $h_{123}[n]$ in this expression yields

$$h[n] = (h_1[n] + h_2[n]) * h_3[n] - h_4[n],$$

as shown in Fig. 2.21(c).

Now substitute the specified forms of $h_1[n]$ and $h_2[n]$ to obtain

$$\begin{aligned} h_{12}[n] &= u[n] + u[n+2] - u[n] \\ &= u[n+2]. \end{aligned}$$

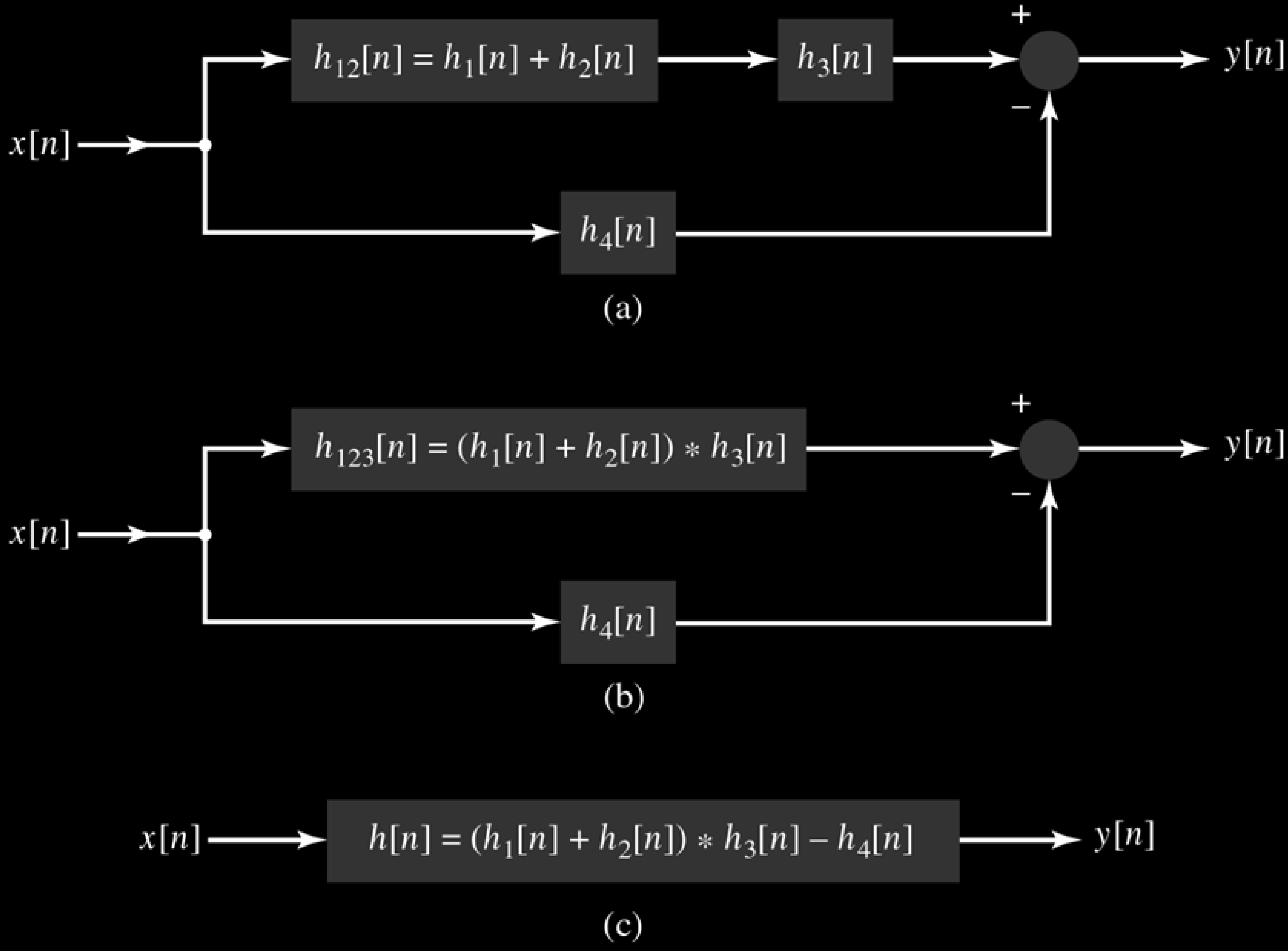
Convolving $h_{12}[n]$ with $h_3[n]$ gives

$$\begin{aligned} h_{123}[n] &= u[n+2] * \delta[n-2] \\ &= u[n]. \end{aligned}$$

Finally, we sum $h_{123}[n]$ and $-h_4[n]$ to obtain the overall impulse response:

$$h[n] = \{1 - \alpha^n\}u[n]. \quad \blacksquare$$

Figure 2.21 (p. 131)
(a) Reduction of parallel combination of LTI systems in upper branch of Fig. 2.20.
(b) Reduction of cascade of systems in upper branch of Fig. 2.21(a).
(c) Reduction of parallel combination of systems in Fig. 2.21(b) to obtain an equivalent system for Fig. 2.20.



Problem 2.8

Find the expression for the impulse response relating the input $x(t)$ to the output $y(t)$ for the system depicted in Fig. 2.22.

Answer: $h(t) = [h_1(t) * h_4(t) - h_2(t) - h_3(t)] * h_5(t)$

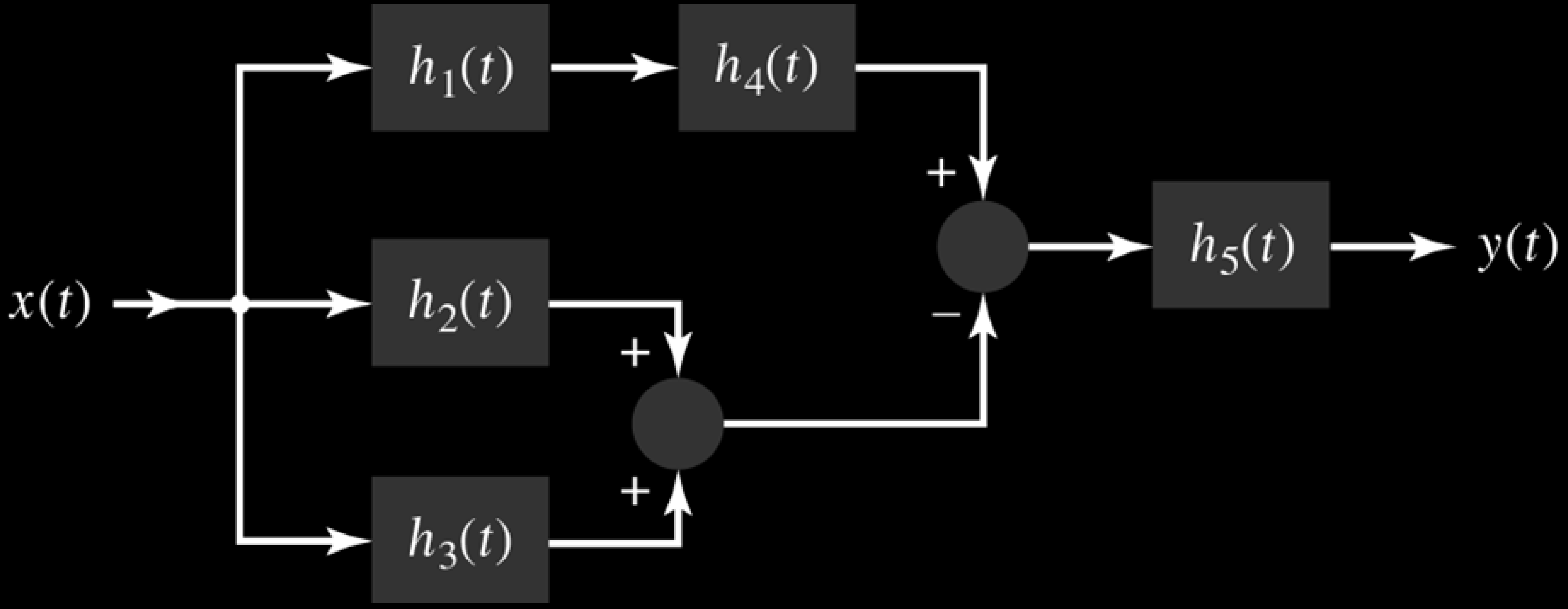


Figure 2.22 (p. 132)
Interconnection of LTI systems for Problem 2.8.

Table 2.1 Interconnection Properties for LTI Systems.

Property	*Continuous-time system*
Distributive	$x(t) * h_1(t) + x(t) * h_2(t) = x(t) * \{h_1(t) + h_2(t)\}$
Associative	$\{x(t) * h_1(t)\} * h_2(t) = x(t) * \{h_1(t) * h_2(t)\}$
Commutative	$h_1(t) * h_2(t) = h_2(t) * h_1(t)$

TABLE 2.1 *Interconnection Properties for LTI Systems.*

Property	*Discrete-time system*
Distributive	$x[n] * h_1[n] + x[n] * h_2[n] =$ $x[n] * \{h_1[n] + h_2[n]\}$
Associative	$\{x[n] * h_1[n]\} * h_2[n] = x[n] * \{h_1[n] * h_2[n]\}$
Commutative	$h_1[n] * h_2[n] = h_2[n] * h_1[n]$

2.7 Relations between LTI System Properties and the Impulse Response

- Properties of the LTI system, such as memory, causality, and stability, are related to the system's impulse response. Here, we explore these…

2.7.1 Memoryless LTI Systems

- The output of a memoryless LTI system depends only on the current input.
- Exploiting the commutative property of convolution, we may express the output of a discrete-time LTI system as:

$$y[n] = h[n] * x[n] = \sum_{k=-\infty}^{\infty} h[k]x[n-k]$$

- Expanding, we have…

$$y[n] = \cdots + h[-2]x[n+2] + h[-1]x[n+1]$$
$$+h[0]x[n] + h[1]x[n-1] + h[2]x[n-2] \qquad (2.27)$$

- For this system to be memoryless, $y[n]$ must depend only on $x[n]$ and therefore cannot depend on $x[n - k]$ for $k \neq 0$.
- Thus a discrete-time LTI system is memoryless iff $h[k] = c\delta[k]$, c is constant.
- For continuous-time system, $h(\tau) = c\delta(\tau)$

2.7.2 Causal LTI Systems

- Output of a causal LTI system depends only on past or present values of the input.

$$y[n] = \cdots + h[-2]x[n+2] + h[-1]x[n+1]$$
$$+h[0]x[n] + h[1]x[n-1] + h[2]x[n-2] \quad (2.27)$$

- For this system to be memoryless, $y[n]$ must depend only on $x[n]$ and therefore
- In order, then, for $y[n]$ to depend only on past or present values of the input, we require that $h[k] = 0$ for $k < 0$.
- And the convolution sum becomes

$$y[n] = \sum_{k=-\infty}^{\infty} h[k]x[n-k]$$

- For continus time systems

$$y(t) = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau$$

2.7.3 Stable LTI Systems

- We recall from Section 1.8.1 that a system is bounded input-bounded output (BIBO) stable if the output is guaranteed to be bounded for every bounded input. Formally, the conditions on $h[n]$ that guarantee stability is given by

$$|y[n]| = |h[n] * x[n]| y[n] = \sum_{k=-\infty}^{\infty} h[k]x[n-k]$$

- The impulse response of a stable discrete-time LTI system satisfies

$$\sum_{k=-\infty}^{\infty} |h[k]| < \infty$$

- and for continuous systems,

$$y(t) = \int_{-\infty}^{\infty} |h(\tau)d\tau| < \infty$$

▶ **Problem 2.10** For each of the following impulse responses, determine whether the corresponding system is (i) memoryless, (ii) causal, and (iii) stable. Justify your answers.

(a) $h(t) = u(t+1) - u(t-1)$

(b) $h(t) = u(t) - 2u(t-1)$

(c) $h(t) = e^{-2|t|}$

(d) $h(t) = e^{at}u(t)$

(e) $h[n] = 2^n u[-n]$

(f) $h[n] = e^{2n}u[n-1]$

(g) $h[n] = (1/2)^n u[n]$

Answers:

(a) not memoryless, not causal, stable.

(b) not memoryless, causal, not stable.

(c) not memoryless, not causal, stable.

(d) not memoryless, causal, stable provided that $a < 0$.

(e) not memoryless, not causal, stable.

(f) not memoryless, causal, not stable.

(g) not memoryless, causal, stable.

2.7.4 Invertible Systems and Deconvolution

- A system is invertible if the input to the system can be recovered from the output except for a constant scale factor
- The process of recovering $x(t)$ from $h(t)*x(t)$ is termed deconvolution, since it corresponds to reversing or undoing the convolution operation
- An inverse system performs deconvolution as shown in Fig. 2.24.
- Equalization of the distortion introduced by over telephone lines is an example
- Equalizer reverses distortion and permits much higher data rates to be achieved
- To derive the relationship between the impulse response of an LTI system, $h(t)$, and that of the corresponding inverse system, $h^{inv}(t)$, we have

$$x(t) * \left(h(t) * h^{inv}(t) \right) = x(t) \Rightarrow h(t) * h^{inv}(t) = \delta(t) \qquad (2.30)$$

- For a discrete-time LTI system

$$h[n] * \ h^{inv}[n] = \delta[n] \quad (2.31)$$

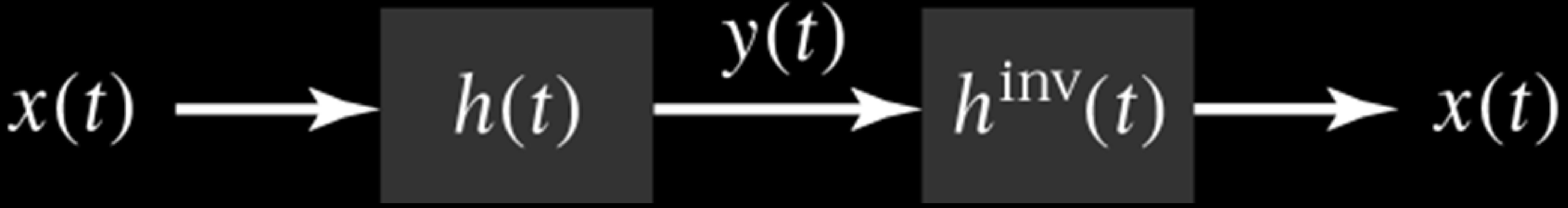
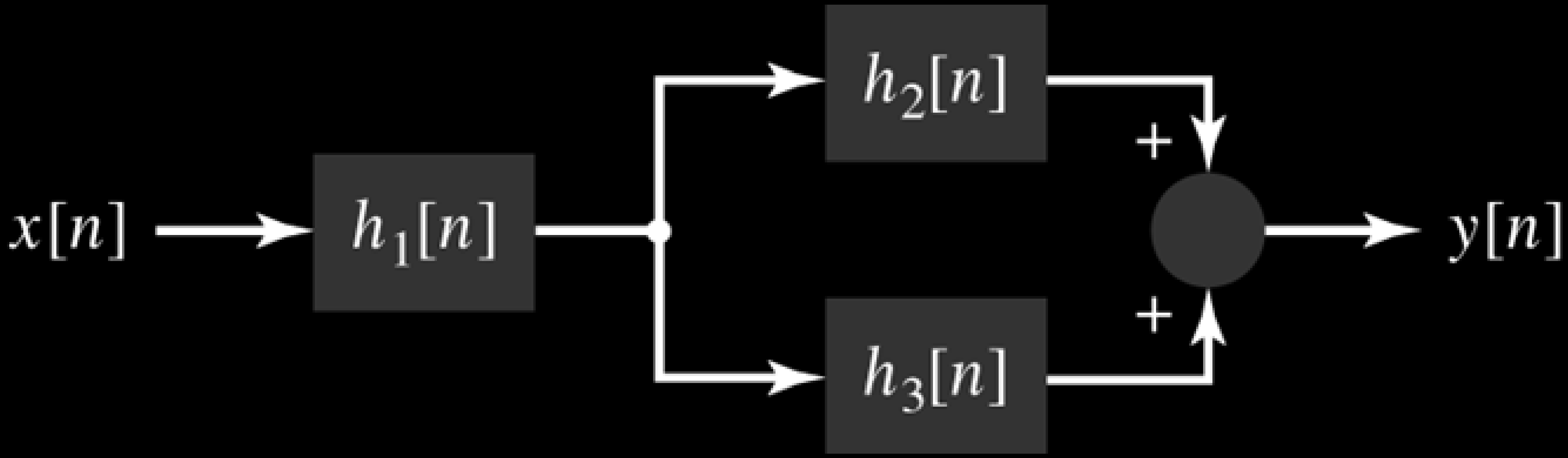


Figure 2.24 (p. 137)
Cascade of LTI system with impulse response $h(t)$ and inverse system with impulse response $h^{-1}(t)$.

Figure 2.23 (p. 132)
Interconnection of LTI systems for Problem 2.9.



EXAMPLE 2.13 MULTIPATH COMMUNICATION CHANNELS: COMPENSATION BY MEANS OF AN INVERSE SYSTEM Consider designing a discrete-time inverse system to eliminate the distortion associated with multipath propagation in a data transmission problem. Recall from Section 1.10 that a discrete-time model for a two-path communication channel is

$$y[n] = x[n] + ax[n-1].$$

Find a causal inverse system that recovers $x[n]$ from $y[n]$. Check whether this inverse system is stable.

Solution: First we identify the impulse response of the system relating $y[n]$ and $x[n]$. We apply an impulse input $x[n] = \delta[n]$ to obtain the impulse response

$$h[n] = \begin{cases} 1, & n = 0 \\ a, & n = 1 \\ 0, & \text{otherwise} \end{cases}$$

as the impulse response of the multipath channel. The inverse system $h^{\text{inv}}[n]$ must satisfy $h[n] * h^{\text{inv}}[n] = \delta[n]$. Substituting for $h[n]$, we see that $h^{\text{inv}}[n]$ must satisfy the equation

$$h^{\text{inv}}[n] + ah^{\text{inv}}[n-1] = \delta[n]. \quad (2.32)$$

Let us solve this equation for several different values of n. For $n < 0$, we must have $h^{\text{inv}}[n] = 0$ in order to obtain a causal inverse system. For $n = 0$, $\delta[n] = 1$, and Eq. (2.32) gives

$$h^{\text{inv}}[0] + ah^{\text{inv}}[-1] = 1.$$

Since causality implies that $h^{\text{inv}}[-1] = 0$, we find that $h^{\text{inv}}[0] = 1$. For $n > 0$, $\delta[n] = 0$, and Eq. (2.32) implies that

$$h^{\text{inv}}[n] + ah^{\text{inv}}[n-1] = 0,$$

which may be rewritten as

$$h^{\text{inv}}[n] = -ah^{\text{inv}}[n-1]. \tag{2.33}$$

Since $h^{\text{inv}}[0] = 1$, Eq. (2.33) implies that $h^{\text{inv}}[1] = -a$, $h^{\text{inv}}[2] = a^2$, $h^{\text{inv}}[3] = -a^3$, and so on. Hence, the inverse system has the impulse response

$$h^{\text{inv}}[n] = (-a)^n u[n].$$

To check for stability, we determine whether $h^{\text{inv}}[n]$ is absolutely summable, which will be the case if

$$\sum_{k=-\infty}^{\infty} |h^{\text{inv}}[k]| = \sum_{k=0}^{\infty} |a|^k$$

is finite. This geometric series converges; hence, the system is stable, provided that $|a| < 1$. This implies that the inverse system is stable if the multipath component $ax[n-1]$ is weaker than the first component $x[n]$; otherwise the system is unstable. ■

TABLE 2.2 ***Properties of the Impulse Response Representation for LTI Sys***

Property	*Continuous-time system*	*Discrete-time system*
Memoryless	$h(t) = c\delta(t)$	$h[n] = c\delta[n]$
Causal	$h(t) = 0$ for $t < 0$	$h[n] = 0$ for $n < 0$
Stability	$\int_{-\infty}^{\infty} \lvert h(t)\rvert \, dt < \infty$	$\sum_{n=-\infty}^{\infty} \lvert h[n]\rvert < \infty$
Invertibility	$h(t) * h^{\text{inv}}(t) = \delta(t)$	$h[n] * h^{\text{inv}}[n] = \delta[n]$

2.8 Step Response

- Step input signals are often used to characterize the response of an LTI system to sudden changes in the input
- $s[n] = h[n] * u[n] = \sum_{k=-\infty}^{\infty} h[k]u[n-k] = \sum_{k=-\infty}^{n} h[k]$

since $u[n\text{-}k] = 0$ for $k > n$

- Similarly, for continuous-time systems, we have

$$s(t) = \int_{-\infty}^{t} h(\tau)d\tau \tag{2.34}$$

- Example 2.14 RC Circuit: Step Response As shown in Example 1.21, the impulse response of the RC circuit depicted in Fig. 2.12 is

$$h(t) = \frac{1}{RC} e^{-\frac{t}{RC}} u(t)$$

- Find the step response of the circuit.
- Solution: Step represents a switch turning on a constant voltage source at $t = 0$. Capacitor voltage increases toward the value of source in an exponential manner. Applying (2.34), we obtain

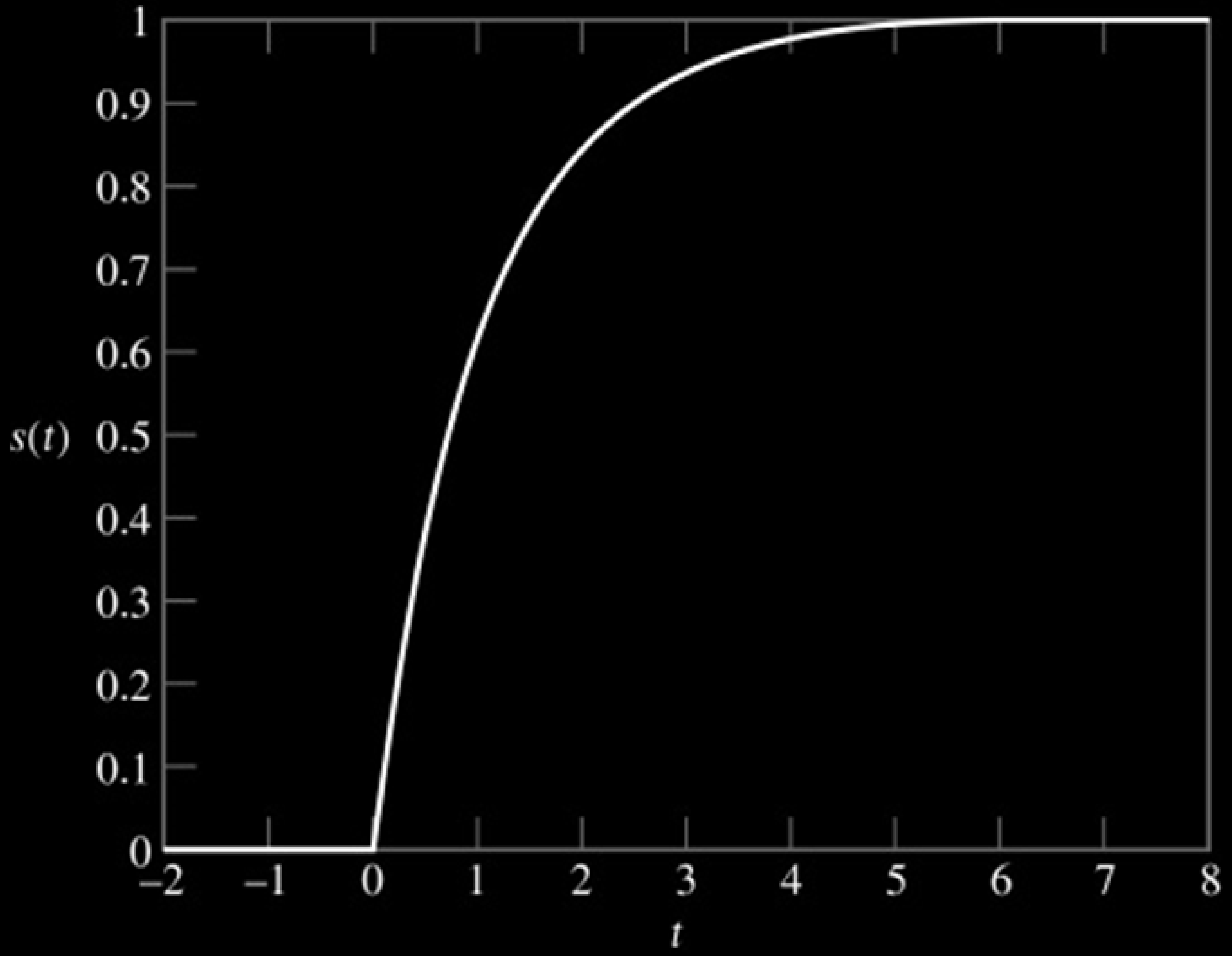
$$s(t) = \int_{-\infty}^{t} \frac{1}{RC} e^{-\frac{\tau}{RC}} u(\tau)\, d\tau$$

- Simplifying, we get:

$$s(t) = \begin{cases} 0, & t < 0 \\ \frac{1}{RC}\int_0^t e^{-\frac{\tau}{RC}}\, d\tau, & t \geq 0 \end{cases} = \begin{cases} 0, & t < 0 \\ 1 - e^{-\frac{t}{RC}}, & t \geq 0 \end{cases}$$

- Figure 2.25 depicts the RC circuit step response for RC = 1 s.

Figure 2.25 (p. 140)
RC circuit step response for *RC* = 1 s.



2.9 Differential and Difference Equation Representations of LTl Systems

- Difference equations are used to represent discrete-time systems, while differential equations represent continuous-time systems. The general form of a linear constant-coefficient differential equation is

$$\sum_{k=0}^{N} a_k \frac{d^k}{dt^k} y(t) = \sum_{k=0}^{M} b_k \frac{d^k}{dt^k} x(t) \qquad (2.35)$$

- where a_k and b_k are constants, $x(t)$ is input and $y(t)$ is the output.
- A linear constant-coefficient difference equation with derivatives replaced by delayed values of the input $x[n]$ and output $y[n]$:

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k] \qquad (2.36)$$

- As an example, consider the RLC circuit in Fig. 2.26 with input voltage source $x(t)$ and the output current, $y(t)$, then summing voltage drops around the loop gives

$$Ry(t) + L\frac{d}{dt}y(t) + \frac{1}{C}\int_{-\infty}^{t} y(\tau)\,d\tau = x(t).$$

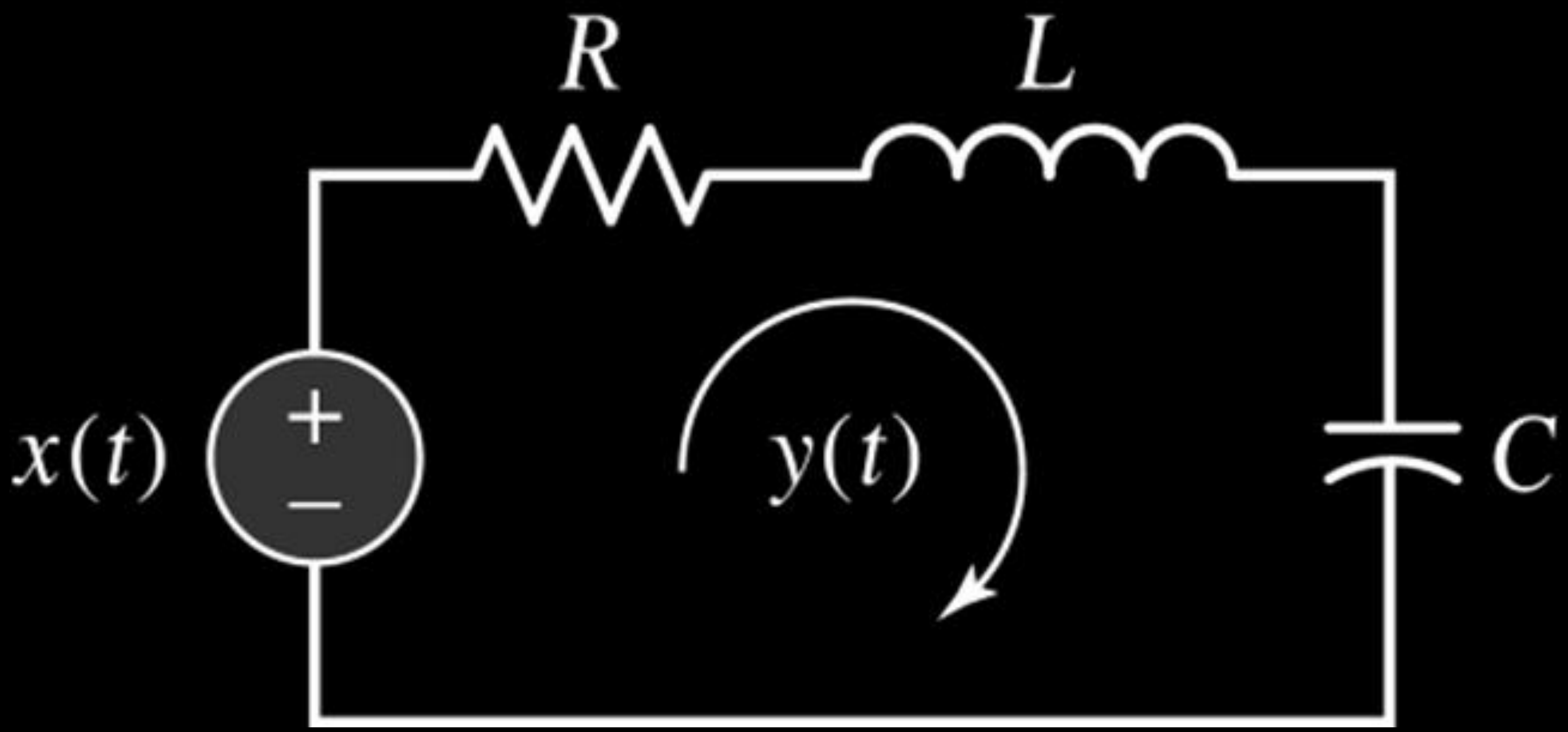
Differentiating both sides of this equation with respect to t results in

$$\frac{1}{C}y(t) + R\frac{d}{dt}y(t) + L\frac{d^2}{dt^2}y(t) = \frac{d}{dt}x(t).$$

- Difference equations are easily rearranged to obtain recursive formulas for computing cu ... om (2.36):

$$y[n] = \frac{1}{a_0}\sum_{k=0}^{M} b_k x[n-k] - \frac{1}{a_0}\sum_{k=1}^{N} a_k y[n-k].$$

Figure 2.26 (p. 141)
Example of an *RLC* circuit described by a differential equation.



- Computing $y[n]$ for $n > 0$ from $x[n]$ for second-order difference equation (2.37),

$$y[n] = x[n] + 2x[n-1] - y[n-1] - \frac{1}{4}y[n-2]. \quad (2.38)$$

- Begin with n = 0, we may determine output by evaluating sequence of equations

$$y[0] = x[0] + 2x[-1] - y[-1] - \frac{1}{4}y[-2],$$

$$y[1] = x[1] + 2x[0] - y[0] - \frac{1}{4}y[-1],$$

$$y[2] = x[2] + 2x[1] - y[1] - \frac{1}{4}y[0],$$

$$y[3] = x[3] + 2x[2] - y[2] - \frac{1}{4}y[1],$$

- Output computed from input and past values of output. To begin calculation at time n=0, we must know 2 recent past values of output, y[-1], y[-2], known as initial conditions

EXAMPLE 2.15 RECURSIVE EVALUATION OF A DIFFERENCE EQUATION Find the first two output values $y[0]$ and $y[1]$ for the system described by Eq. (2.38), assuming that the input is $x[n] = (1/2)^n u[n]$ and the initial conditions are $y[-1] = 1$ and $y[-2] = -2$.

Solution: Substitute the appropriate values into Eq. (2.39) to obtain

$$y[0] = 1 + 2 \times 0 - 1 - \frac{1}{4} \times (-2) = \frac{1}{2}.$$

Now substitute for $y[0]$ in Eq. (2.40) to find

$$y[1] = \frac{1}{2} + 2 \times 1 - \frac{1}{2} - \frac{1}{4} \times (1) = 1\frac{3}{4}.$$ ■

▶ **Problem 2.14** Write a differential equation describing the relationship between the input voltage $x(t)$ and current $y(t)$ through the inductor in Fig. 2.29.

Answer:

$$Ry(t) + L\frac{d}{dt}y(t) = x(t).$$ ◀

▶ **Problem 2.15** Calculate $y[n]$, $n = 0, 1, 2, 3$ for the first-order recursive system

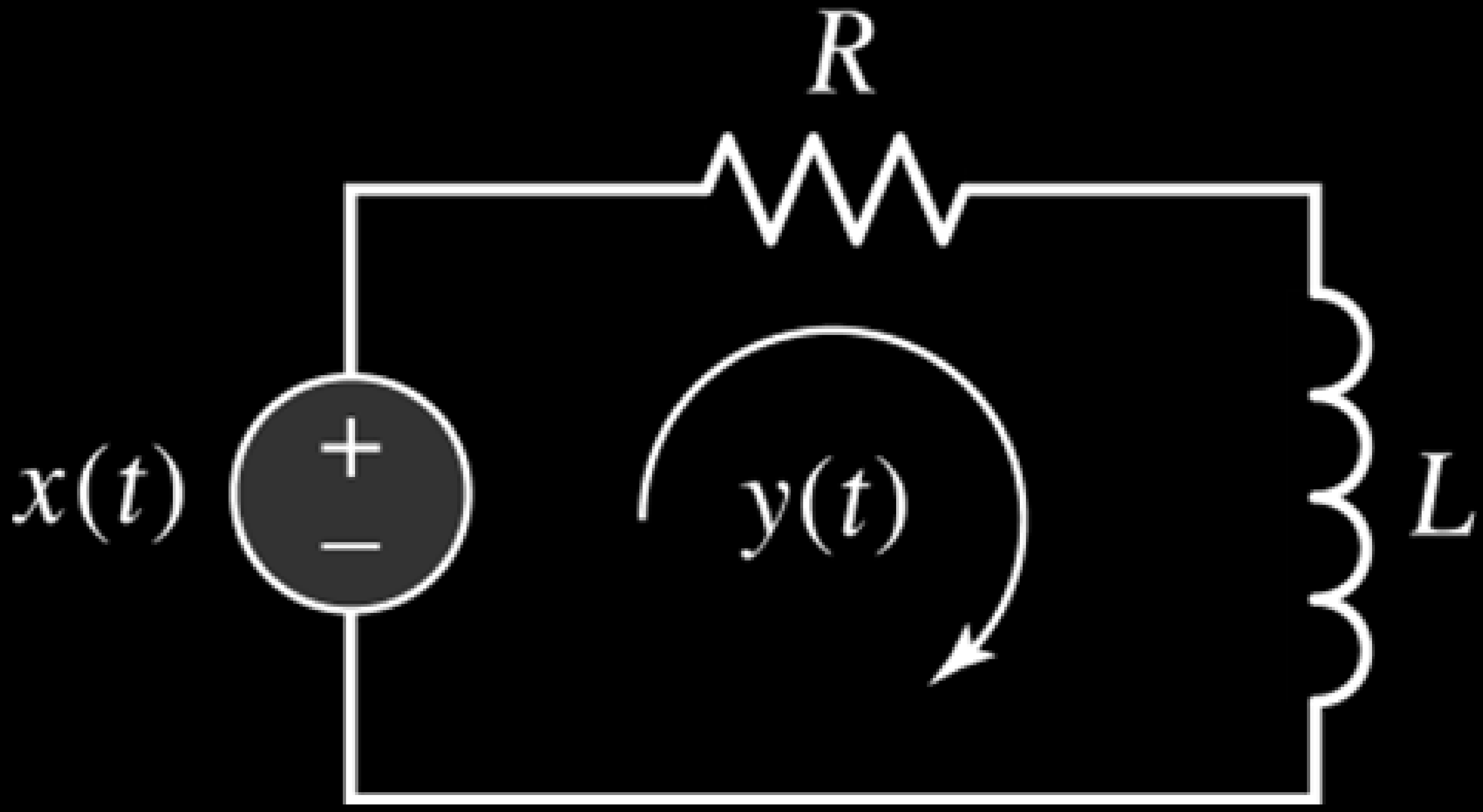
$$y[n] - (1/2)y[n-1] = x[n]$$

if the input is $x[n] = u[n]$ and the initial condition is $y[-1] = -2$.

Answer:

$$y[0] = 0, \quad y[1] = 1, \quad y[2] = 3/2, \quad y[3] = 7/4.$$ ◀

Figure 2.29 (p. 146): RL circuit.



2.10 Solving Differential and Difference Equations

- The output of a system described by a differential or difference equation may be expressed as the sum of two components.
  - 1st is solution of differential or difference equation, called **homogeneous solution** denote by $y^{(h)}$
  - 2nd is solution of the original equation, which we term as particular solution and denote by $y^{(p)}$
- Thus, the complete solution is $y = y^{(h)} + y^{(p)}$, with the arguments t or n omitted

2.10.1 The Homogeneous Solution

- The homogeneous form of a differential or difference equation is obtained by setting all terms involving the input to zero and solving for the homogeneous equation for $y^{(h)}(t)$,

$$\sum_{k=0}^{N} a_k \frac{d^k}{dt^k} y^{(h)}(t) = 0$$

2.10.2 The Particular Solution

- The particular solution $y^{(p)}$ represents any solution of differential or difference equation for given input.
- If input is $x[n] = A\cos(\Omega n + \phi)$, we assume general sinusoidal response of form

$$y^{(p)}[n] = c_1 \cos(\Omega n) + c_1 \sin(\Omega n),$$

- where c_1 and c_2 are determined from $y^{(p)}[n]$

2.10.3 The Complete Solution

- The complete solution of the differential or difference equation is obtained by summing the particular solution and the homogeneous solution
- finding the unspecified coefficients in the homogeneous solution so that the complete solution satisfies the prescribed initial conditions

2.13 State-Variable Descriptions of LTl Systems

- The state of a system may be defined as a minimal set of signals that represent the system's entire memory of the past.
- That is, given only the value of the state at an initial point in time, n_i, (or t_i), and the input for times , $n > n_i$, (or $t > t_i$), we can determine the output for all times $n > n_i$, (or $t > t_i$)
- There are many possible state-variable descriptions corresponding to a system with a given input-output characteristic.

2.13.1 The State-Variable Description

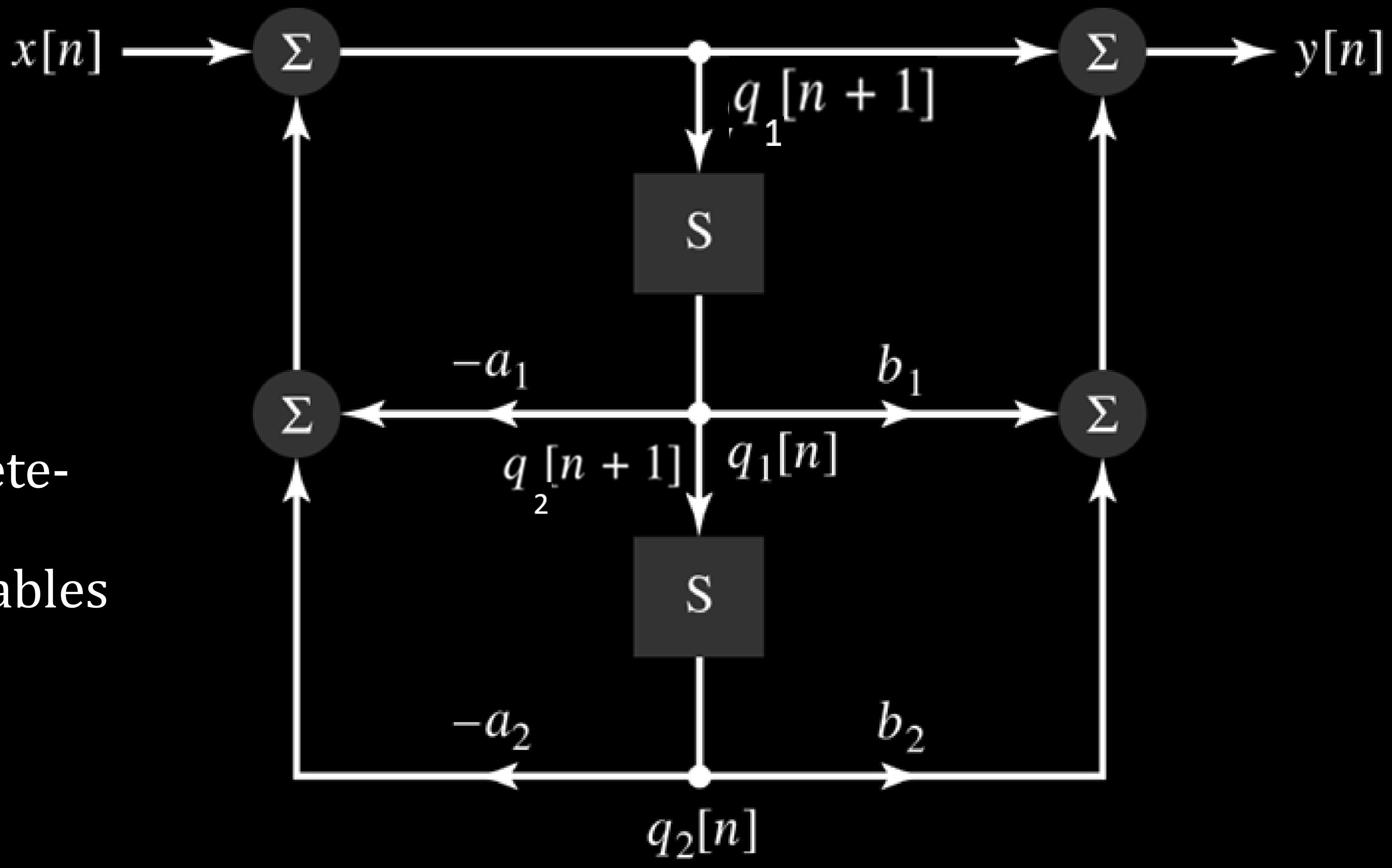
- We shall develop the general state-variable description by starting with the direct form II implementation of a second-order LTI system, depicted in Fig. 2.39.
- In order to determine output of the system for $n \geq n_i$, we need the input for $n \geq n_i$, and the outputs of the time-shift operations labeled $q_1[n]$ and $q_2[n]$ at time $n = n_i$.
- This suggests that we may choose $q_1[n]$ and $q_2[n]$ as the state of the system.
- Since $q_1[n]$ and $q_2[n]$ are outputs of time-shift operations, next value of the state, $q_1[n + 1]$ and $q_2[n + 1]$, must correspond to variables at input to time-shift operations
- The block diagram indicates that the next value of the state is obtained from the current state and the input via the two equations

$$q_1[n+1] = -a_1 q_1[n] + -a_2 q_2[n] + x[n] \tag{2.57}$$

- and

$$q_2[n+1] = q_1[n] \tag{2.58}$$

Figure 2.39 (p. 167)
Direct form II representation of a second-order discrete-time LTI system depicting state variables $q_1[n]$ and $q_2[n]$.



- The block diagram also indicates that the system output is expressed in terms of the input and the state of the system as

$$y[n] = x[n] - \mathrm{a}_1\mathrm{q}_1[n] - a_2q_2[n] + b_1q_1[n] + b_2q_2[n]$$

- or

$$y[n] = (b_1 - \mathrm{a}_1)q_1[n]\ (b_2 - a_2)q_2[n] + x[n] \qquad (2.59)$$

- We write Eqs. (2.57) and (2.58) in matrix form as

$$\begin{bmatrix} q_1[n+1] \\ q_2[n+1] \end{bmatrix} = \begin{bmatrix} -a_1 & -a_2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} q_1[n] \\ q_2[n] \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} x[n] \qquad (2.60)$$

- while Eq. (2.59) is expressed as

$$y[n] = [b_1 - a_1 \quad b_2 - a_2] \begin{bmatrix} q_1[n] \\ q_2[n] \end{bmatrix} + [1]x[n] \qquad (2.61)$$

- If we define the state vector as the column vector

$$\mathbf{q}[n] = \begin{bmatrix} q_1[n] \\ q_2[n] \end{bmatrix}$$

- Then we can rewrite Eqs. (2.60) and (2.61) as

$$q[n+1] = Aq[n] + bx[n] \qquad (2.62)$$

- and

$$y[n] = cq[n] + Dx[n] \qquad (2.63)$$

- where matrix A, vectors b and c, and scalar D are given by

$$\mathbf{A} = \begin{bmatrix} -a_1 & -a_2 \\ 1 & 0 \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

$$\mathbf{c} = [b_1 - a_1 \quad b_2 - a_2], \quad \text{and} \quad D = 1.$$

- Equations (2.62) and (2.63) are general form of a state-variable description corresponding to a discrete-time system
- Matrix A, vectors b and c, and scalar D represent another description of system
- Thus, the state-variable description is used in any problem in which the internal system structure needs to be considered
- If the input-output characteristics of the system are described by an N^{th}-order difference equation, then the state vector $q[n]$ is N by 1, A is N by N, b is N by 1, and c is 1 by N
- Recall that solving of the difference equation requires N initial conditions, which represent the system's memory of the past, as does the N-dimensional state vector
- Also, an N^{th}-order system contains at least N time-shift operations in its block diagram representation.

EXAMPLE 2.28 STATE-VARIABLE DESCRIPTION OF A SECOND-ORDER SYSTEM Find the state-variable description corresponding to the system depicted in Fig. 2.40 by choosing the state variables to be the outputs of the unit delays.

Solution: The block diagram indicates that the states are updated according to the equations

$$q_1[n+1] = \alpha q_1[n] + \delta_1 x[n]$$

and

$$q_2[n+1] = \gamma q_1[n] + \beta q_2[n] + \delta_2 x[n]$$

and the output is given by

$$y[n] = \eta_1 q_1[n] + \eta_2 q_2[n].$$

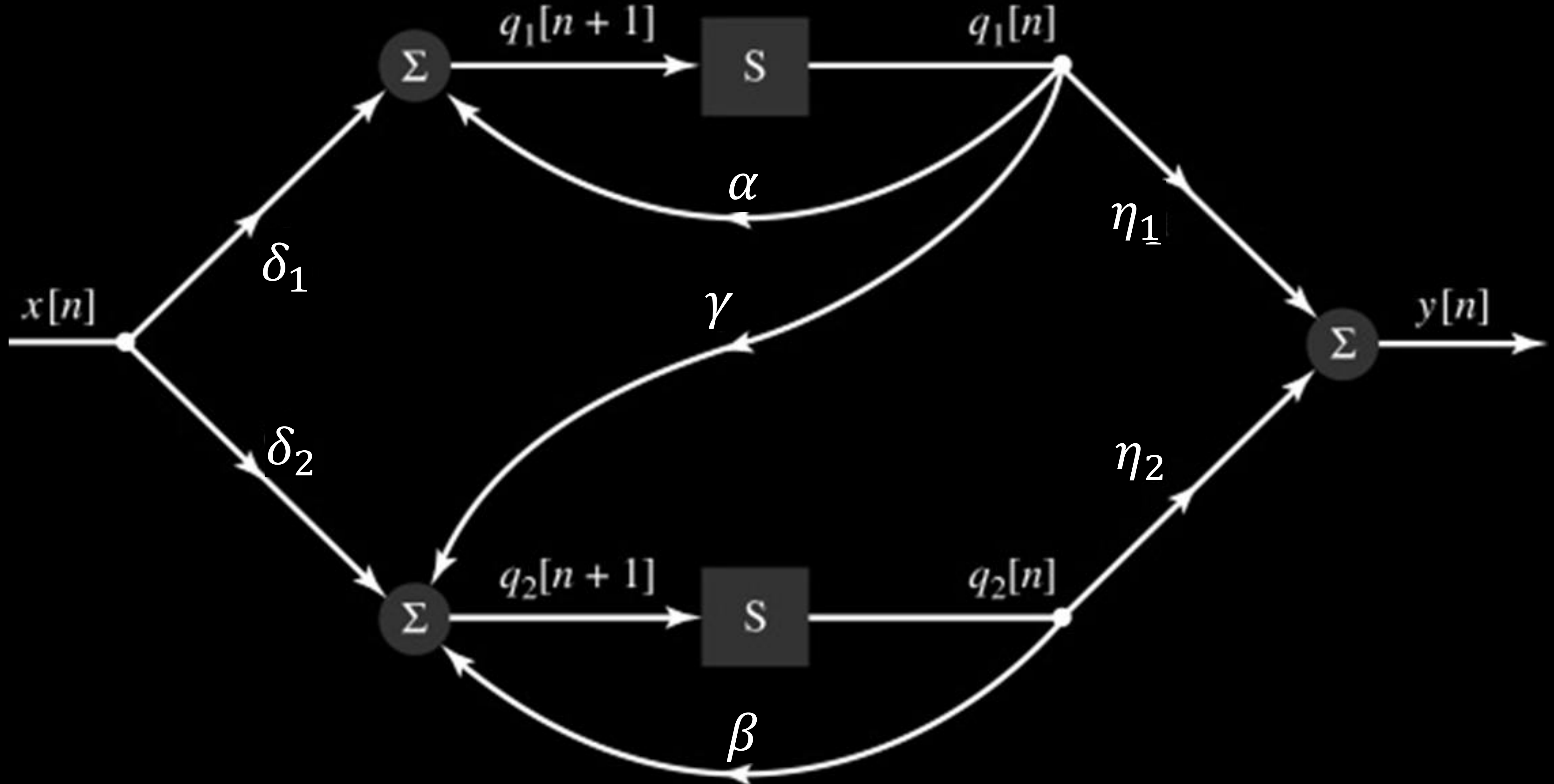
These equations may be expressed in the state-variable forms of Eqs. (2.62) and (2.63) if we define

$$\mathbf{q}[n] = \begin{bmatrix} q_1[n] \\ q_2[n] \end{bmatrix},$$

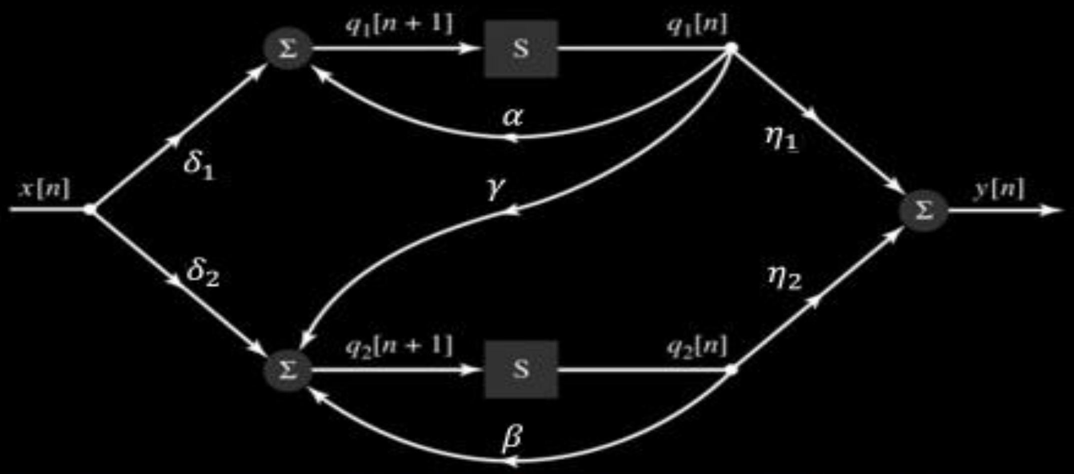
$$\mathbf{A} = \begin{bmatrix} \alpha & 0 \\ \gamma & \beta \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} \delta_1 \\ \delta_2 \end{bmatrix},$$

$$\mathbf{c} = [\eta_1 \quad \eta_2], \quad \text{and} \quad D = [0].$$ ■

Figure 2.40 (p. 169): Block diagram of LTI system for Example 2.28.



Q) Find the state variable description of the system given.



Finding state variable description of a system means writing down the expressions

$$q[n+1] = Aq[n] + bx[n]$$

and

$$y[n] = cq[n] + Dx[n]$$

where

$$q[n+1] = \begin{bmatrix} q_1[n+1] \\ q_2[n+1] \end{bmatrix} \text{ and } q[n] = \begin{bmatrix} q_1[n] \\ q_2[n] \end{bmatrix}$$

$x[n]$ is the input and $y[n]$ is the output.

To find the state variable description of this system we need to find $q_1[n+1]$ and $q_2[n+1]$ and $y[n]$ in terms of $q_1[n]$, $q_2[n]$ and $y[n]$. Lets do it together…

$$q_1[n+1] = \alpha q_1[n] + 0q_2[n] + \delta_1 x[n]$$

$$q_2[n+1] = \gamma q_1[n] + \beta q_2[n] + \delta_2 x[n]$$

Combining them into

$$\begin{bmatrix} q_1[n+1] \\ q_2[n+1] \end{bmatrix} = \begin{bmatrix} \alpha & 0 \\ \gamma & \beta \end{bmatrix} \begin{bmatrix} q_1[n] \\ q_2[n] \end{bmatrix} + \begin{bmatrix} \delta_1 \\ \delta_2 \end{bmatrix} x[n]$$

$$q[n+1] = \begin{bmatrix} \alpha & 0 \\ \gamma & \beta \end{bmatrix} q[n] + \begin{bmatrix} \delta_1 \\ \delta_2 \end{bmatrix} x[n]$$

Then we need to find the output in terms of $q_1[n]$, $q_2[n]$ and $y[n]$. Lets do it together…

$$y[n] = \eta_1 q_1[n] + \eta_2 q_2[n] + 0x[n]$$

In matrix form, we have

$$y[n] = [\eta_1 \;\; \eta_2] q[n] + 0x[n]$$

Hence, we have

$$A = \begin{bmatrix} \alpha & 0 \\ \gamma & \beta \end{bmatrix}, \quad b = \begin{bmatrix} \delta_1 \\ \delta_2 \end{bmatrix}, \quad c = [\eta_1 \;\; \eta_2] \text{ and}$$

$$D = 0$$

▶ **Problem 2.26** Find the state-variable description corresponding to the block diagram representations in Figs. 2.41(a) and (b). Choose the state variables to be the outputs of the unit delays, $q_1[n]$ and $q_2[n]$, as indicated in the figure.

Answers:

(a)

$$\mathbf{A} = \begin{bmatrix} -\frac{1}{2} & 0 \\ 1 & \frac{1}{3} \end{bmatrix}; \quad \mathbf{b} = \begin{bmatrix} 1 \\ 3 \end{bmatrix};$$

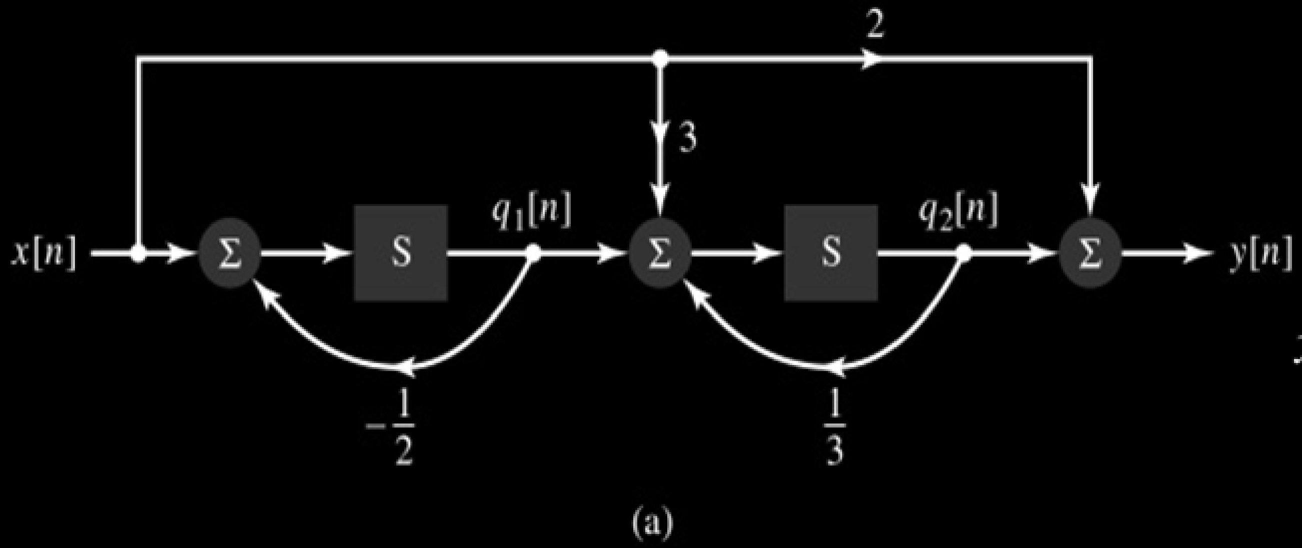
$$\mathbf{c} = [0 \quad 1]; \quad D = [2].$$

(b)

$$\mathbf{A} = \begin{bmatrix} 0 & -\frac{1}{3} \\ \frac{1}{4} & 0 \end{bmatrix}; \quad \mathbf{b} = \begin{bmatrix} 2 \\ -1 \end{bmatrix};$$

$$\mathbf{c} = [1 \quad -2]; \quad D = [0].$$

◀



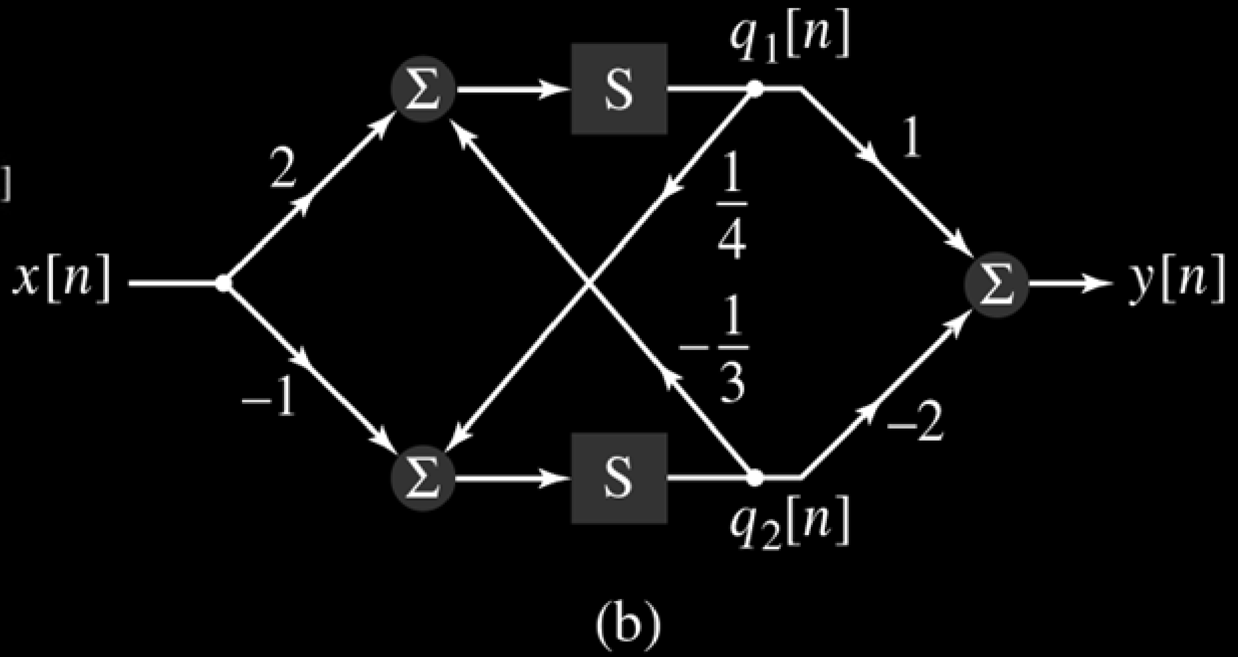


Figure 2.41a (p. 170)
Block diagram of LTI system for Problem 2.26 (2.41b on next slide).

Figure 2.41b (p. 170)

- The state-variable description of continuous-time systems is analogous to that of discrete-time systems, with the exception that the state equation given by (2.62) is expressed in terms of a derivative, i.e.

$$\frac{d}{dt}q(t) = Aq(t) + bx(t) \qquad (2.64)$$

- and

$$y(t) = cq(t) + Dx(t) \qquad (2.64)$$

- Matrix A, vectors b and c, and scalar D describe the internal structure of system
- State variables are usually chosen as the physical quantities associated with such devices.
  - For example, in electrical systems, the energy storage devices are capacitors and inductors.
  - Accordingly, we may choose state variables to correspond to the voltages across capacitors or the currents through inductors

Example 2.29 State-Variable Description of an Electrical Circuit Consider the electrical circuit depicted in Fig. 2.42. Derive a state-variable description of this system if the input is the applied voltage $x(t)$ and the output is the current $y(t)$ through the resistor.

Solution: Choose the state variables as the voltage across each capacitor. Summing the voltage drops around the loop involving $x(t)$, R_1, and C_1 gives

$$x(t) = y(t)R_1 + q_1(t),$$

or

$$y(t) = -\frac{1}{R_1}q_1(t) + \frac{1}{R_1}x(t). \tag{2.66}$$

This equation expresses the output as a function of the state variables and the input $x(t)$. Let $i_2(t)$ be the current through R_2. Summing the voltage drops around the loop involving C_1, R_2, and C_2, we obtain

$$q_1(t) = R_2 i_2(t) + q_2(t),$$

or

$$i_2(t) = \frac{1}{R_2}q_1(t) - \frac{1}{R_2}q_2(t). \tag{2.67}$$

Figure 2.42 (p. 171): Circuit diagram of LTI system for Example 2.29.

Algorithm for Solution

Step 1: KVL in loop 1 to find $y(t)$

Step 2: KVL in loop 2 to find

$i_2(t)$ then $\frac{d}{dt}q_2(t)$

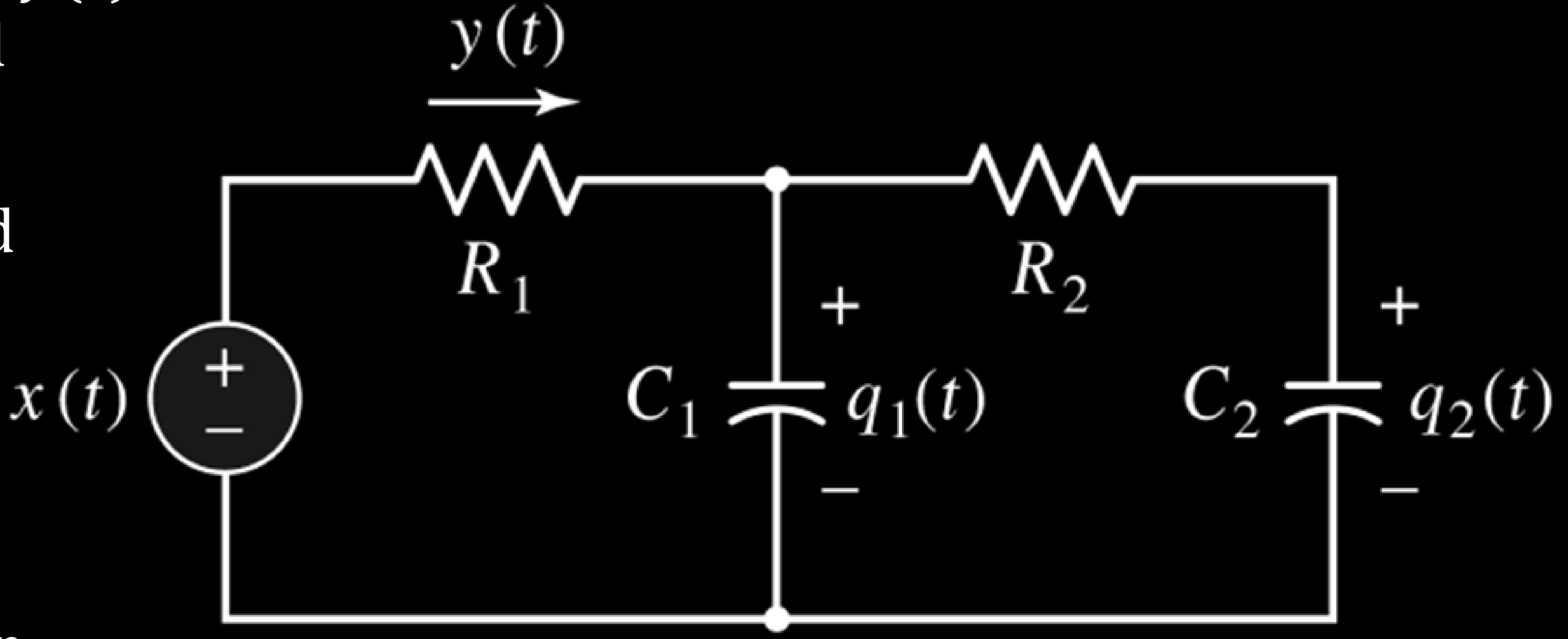
Step 3: KCL at node 1 to find

$$i_1(t) = \frac{d}{dt}q_1(t)$$
$$= y(t) - i_2(t)$$

Step 4: KVL at loop 1

rearranged to obtain

$y(t) = cq(t) + Dx(t)$



However, we also know that

$$i_2(t) = C_2 \frac{d}{dt} q_2(t).$$

We use Eq. (2.67) to eliminate $i_2(t)$ and obtain

$$\frac{d}{dt} q_2(t) = \frac{1}{C_2 R_2} q_1(t) - \frac{1}{C_2 R_2} q_2(t). \tag{2.68}$$

To conclude our derivation, we need a state equation for $q_1(t)$. This is obtained by applying Kirchhoff's current law to the node between R_1 and R_2. Letting $i_1(t)$ be the current through C_1, we have

$$y(t) = i_1(t) + i_2(t).$$

Now we use Eq. (2.66) for $y(t)$, Eq. (2.67) for $i_2(t)$, the relation

$$i_1(t) = C_1 \frac{d}{dt} q_1(t)$$

and rearrange terms to obtain

$$\frac{d}{dt}q_1(t) = -\left(\frac{1}{C_1R_1} + \frac{1}{C_1R_2}\right)q_1(t) + \frac{1}{C_1R_2}q_2(t) + \frac{1}{C_1R_1}x(t). \tag{2.69}$$

The state-variable description, from Eqs. (2.66), (2.68), and (2.69), is

$$\mathbf{A} = \begin{bmatrix} -\left(\frac{1}{C_1R_1} + \frac{1}{C_1R_2}\right) & \frac{1}{C_1R_2} \\ \frac{1}{C_2R_2} & -\frac{1}{C_2R_2} \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} \frac{1}{C_1R_1} \\ 0 \end{bmatrix}$$

$$\mathbf{c} = \begin{bmatrix} -\frac{1}{R_1} & 0 \end{bmatrix}, \text{ and } D = \frac{1}{R_1}.$$

■

► **Problem 2.27** Find the state-variable description of the circuit depicted in Fig. 2.43. Choose the state variables $q_1(t)$ and $q_2(t)$ as the voltage across the capacitor and the current through the inductor, respectively.

Answer:

$$\mathbf{A} = \begin{bmatrix} \dfrac{-1}{(R_1 + R_2)C} & \dfrac{-R_1}{(R_1 + R_2)C} \\ \dfrac{R_1}{(R_1 + R_2)L} & \dfrac{-R_1R_2}{(R_1 + R_2)L} \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} \dfrac{1}{(R_1 + R_2)C} \\ \dfrac{R_2}{(R_1 + R_2)L} \end{bmatrix},$$

$$\mathbf{c} = \begin{bmatrix} \dfrac{-1}{R_1 + R_2} & \dfrac{-R_1}{R_1 + R_2} \end{bmatrix}, \qquad D = \begin{bmatrix} \dfrac{1}{R_1 + R_2} \end{bmatrix}.$$ ◄

In a block diagram representation of a continuous-time system, the state variables correspond to the outputs of the integrators. Thus, the input to the integrator is the derivative of the corresponding state variable. The state-variable description is obtained by writing equations that correspond to the operations in the block diagram. The procedure is illustrated in the next example.

Figure 2.43 (p. 173)
Circuit diagram of LTI system for Problem 2.27.

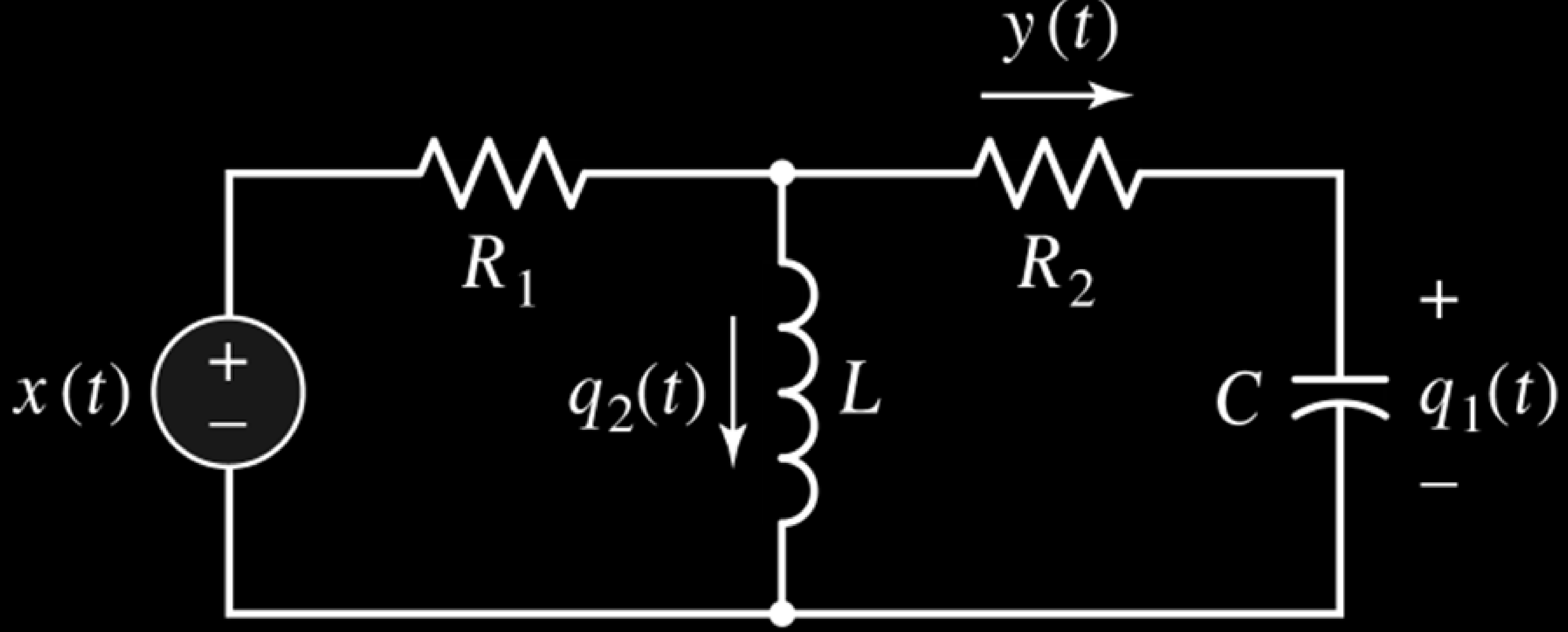
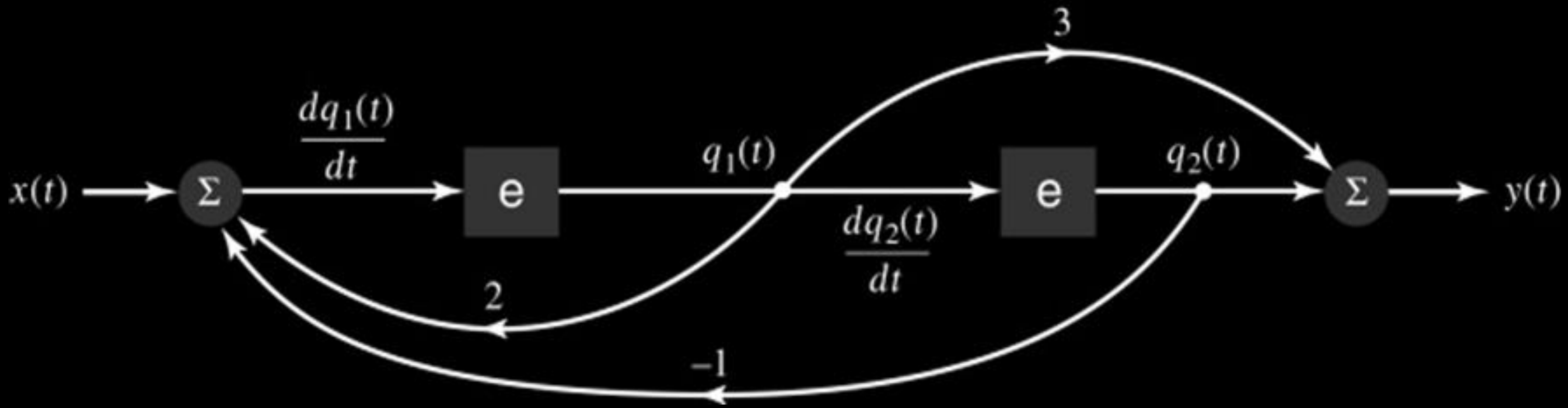


Figure 2.44 (p. 172)

Block diagram of LTI system for Example 2.30.



EXAMPLE 2.30 STATE-VARIABLE DESCRIPTION FROM A BLOCK DIAGRAM Determine the state-variable description corresponding to the block diagram in Fig. 2.44. The choice of state variables is indicated on the diagram.

Solution: The block diagram indicates that

$$\frac{d}{dt}q_1(t) = 2q_1(t) - q_2(t) + x(t),$$

$$\frac{d}{dt}q_2(t) = q_1(t),$$

and

$$y(t) = 3q_1(t) + q_2(t).$$

Hence, the state-variable description is

$$\mathbf{A} = \begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

$$\mathbf{c} = [3 \quad 1], \quad \text{and} \quad D = [0].$$

2.13.2 Transformations of the State

- Different state-variable descriptions are obtained by transforming state variables
- The transformation is accomplished by defining a new set of state variables that are a weighted sum of the original ones
- This changes the form of A, b, c, and D, but does not change the input-output characteristics of the system
- To illustrate the procedure, consider Example 2.30 again. Let us define new states $q_2'(t) = q_1(t)$ and $q_1'(t) = q_2(t)$
- Here, we simply have interchanged the state variables: $q_2'(t)$ is the output of the first integrator and $q_1'(t)$ is the output of the second integrator
- The state-variable description is

$$\mathbf{A}' = \begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix}, \quad \mathbf{b}' = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

$$\mathbf{c}' = [1 \quad 3], \quad \text{and} \quad D' = [0].$$

End of Chapter 2

Please prepare for the next quiz on Chapter 2…

Figure 2.45 (p. 177)

Convolution sum computed using MATLAB.

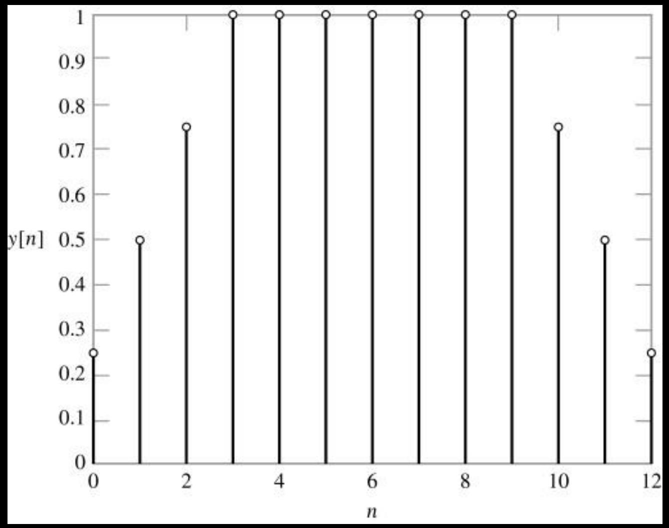


Figure 2.46 (p. 177)

Solution to Problem 2.29.

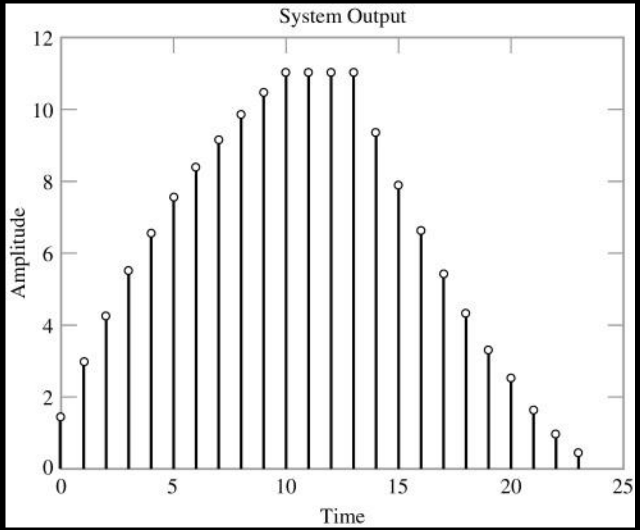


Figure 2.47 (p. 178)

Step response computed using MATLAB.

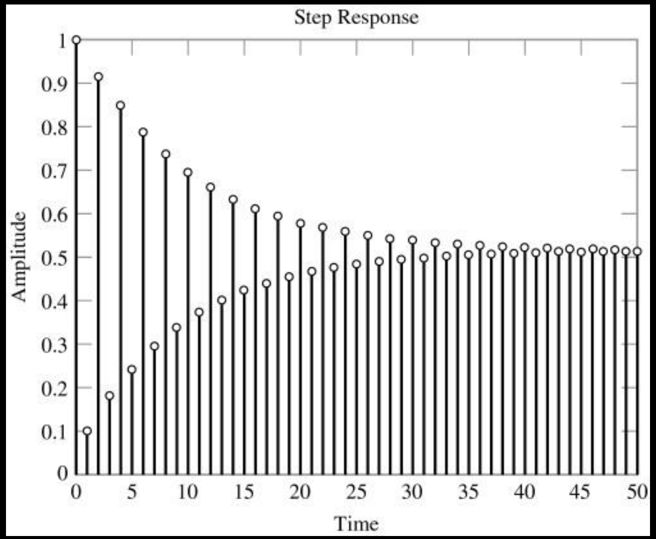


Figure 2.48 (p. 181)

Impulse responses associated with the original and transformed state-variable descriptions computer using MATLAB.

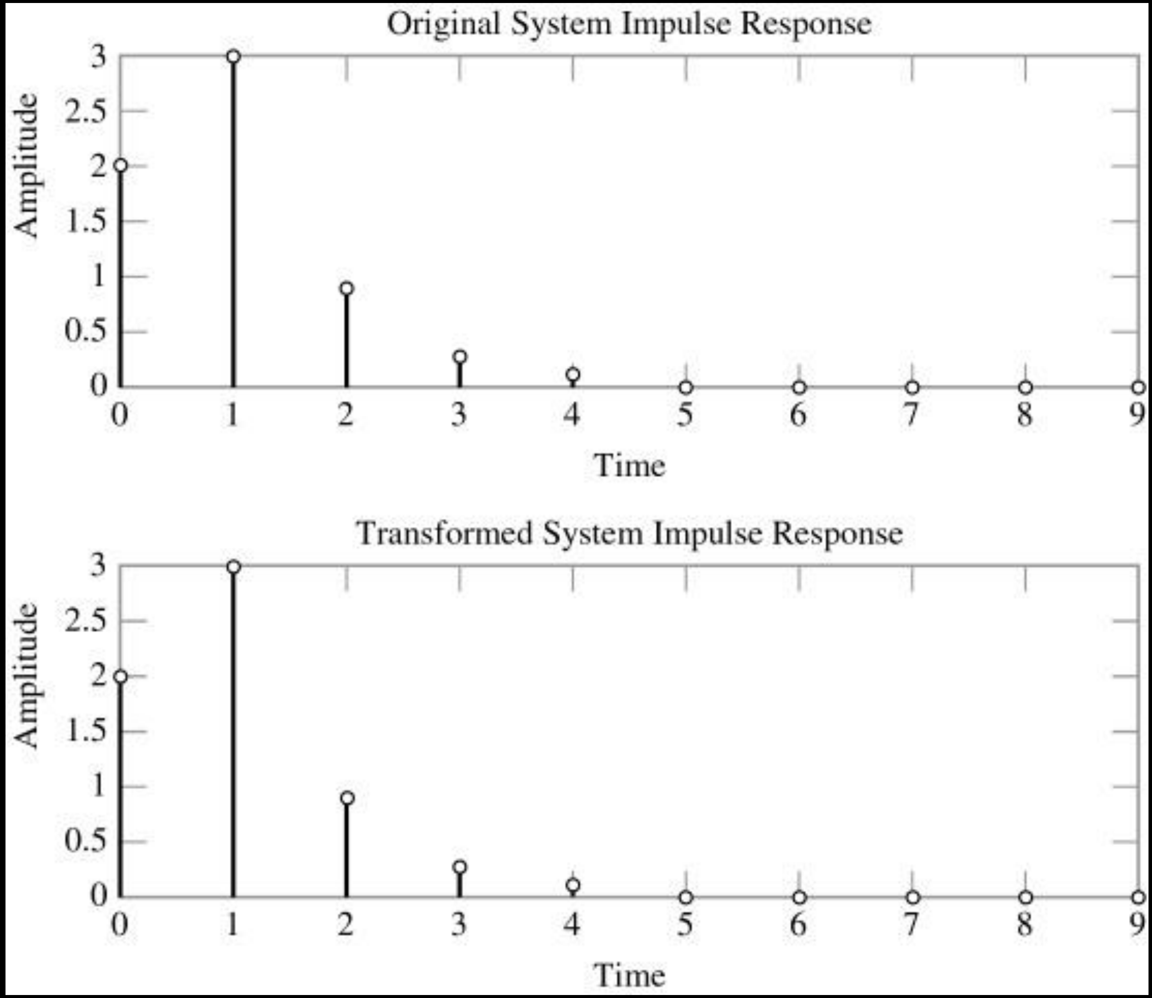


Figure P2.32 (p. 183)

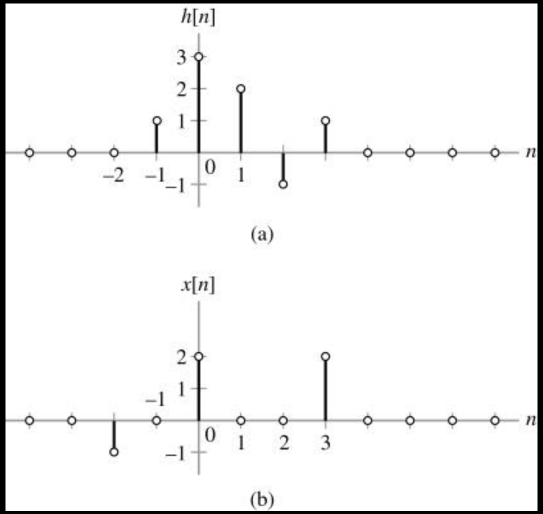


Figure P2.34 (p. 184)

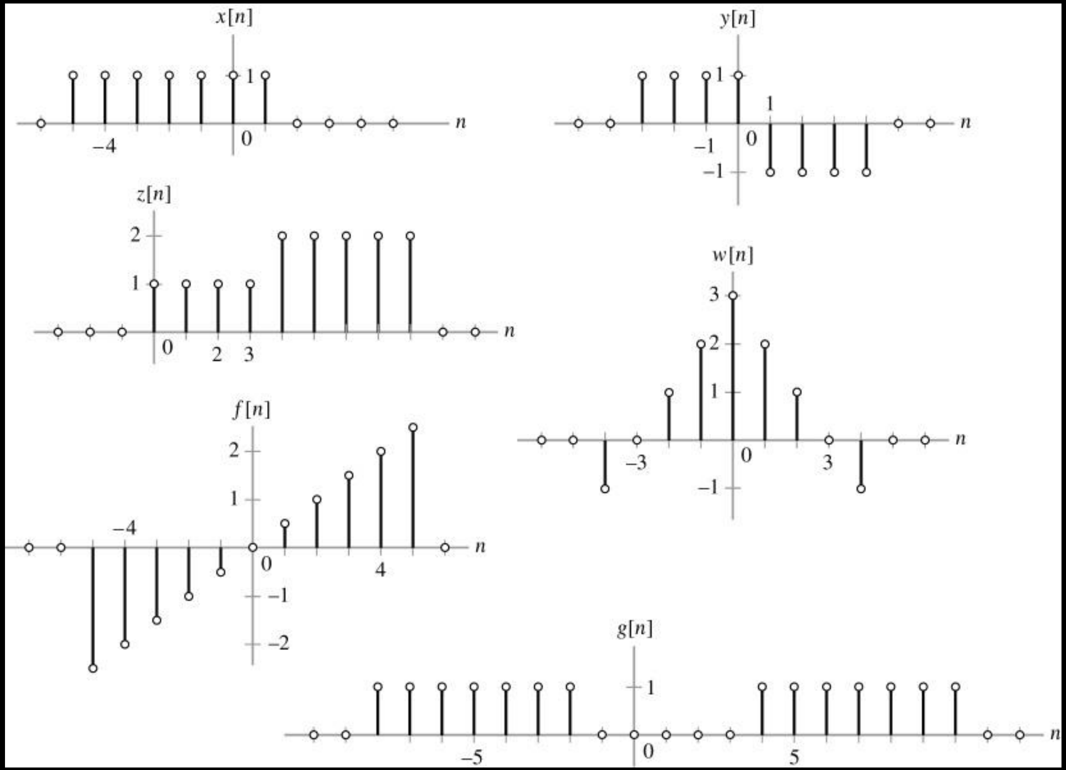


Figure P2.38 (p. 185)

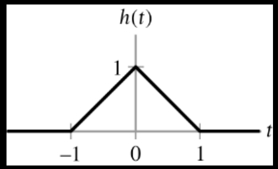


Figure P2.40 (p. 186)

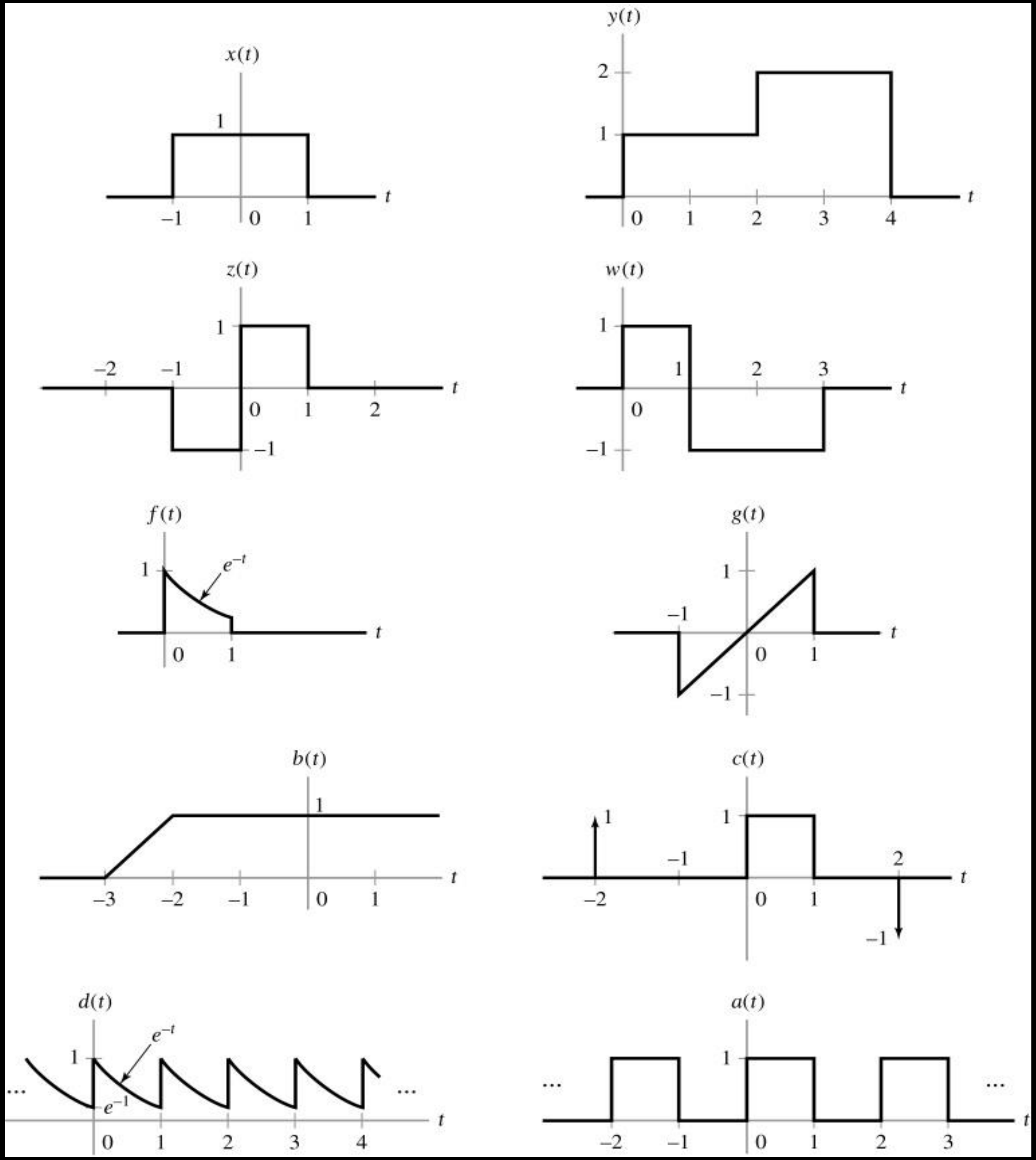


Figure P2.41 (p. 187)

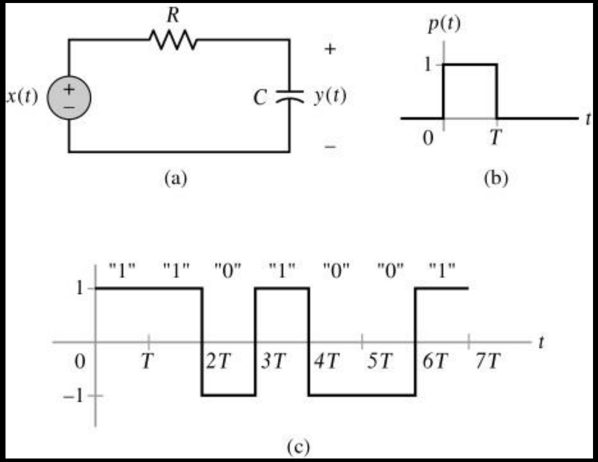


Figure P2.43 (p. 187)

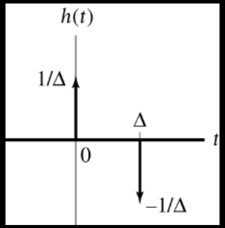
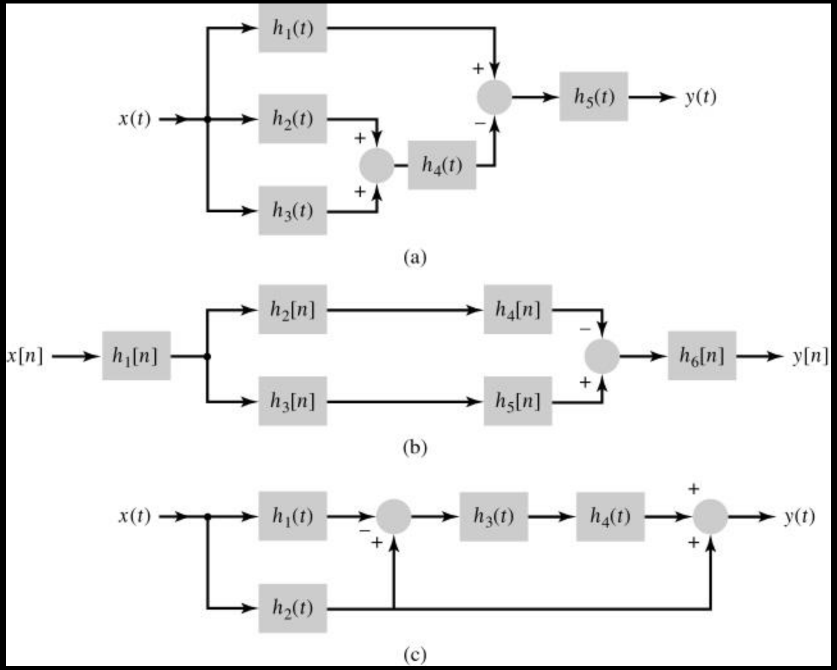
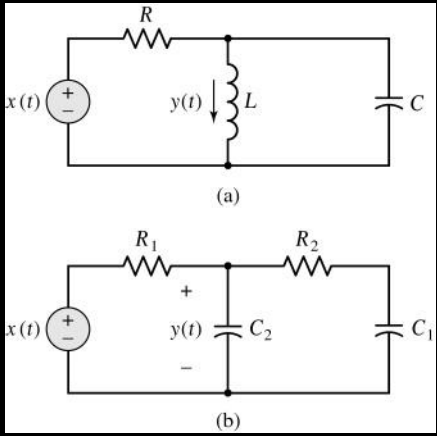


Figure P2.46 (p. 188)





(a)
(b)

Figure P2.61 (p. 189)

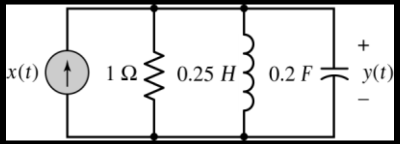


Figure P2.65 (p. 1

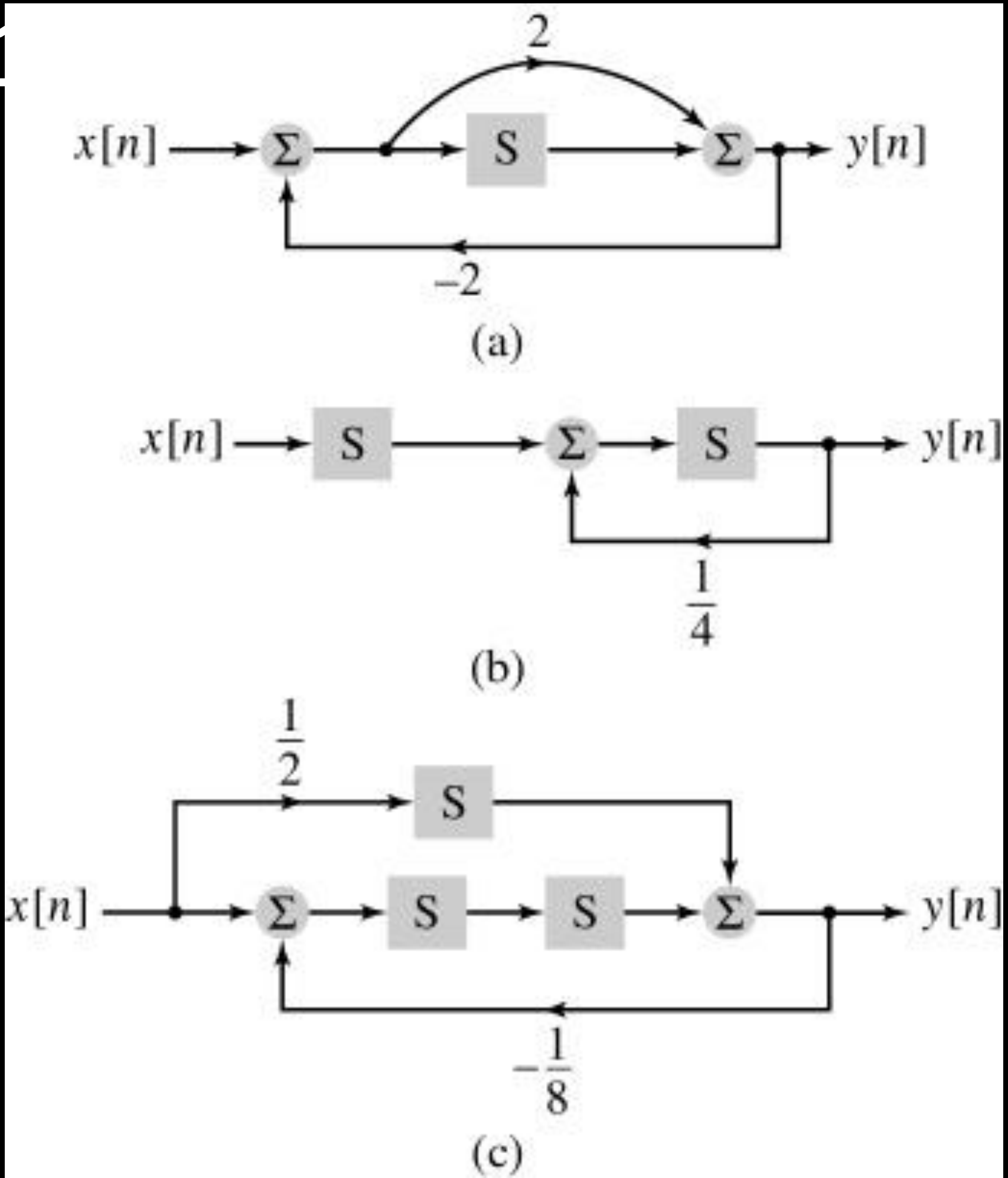
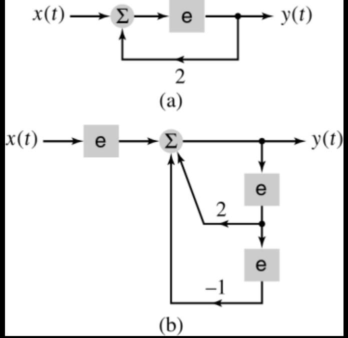


Figure P2.68 (p. 190)



(a)
(b)

Figure P2.69 (p. 190)

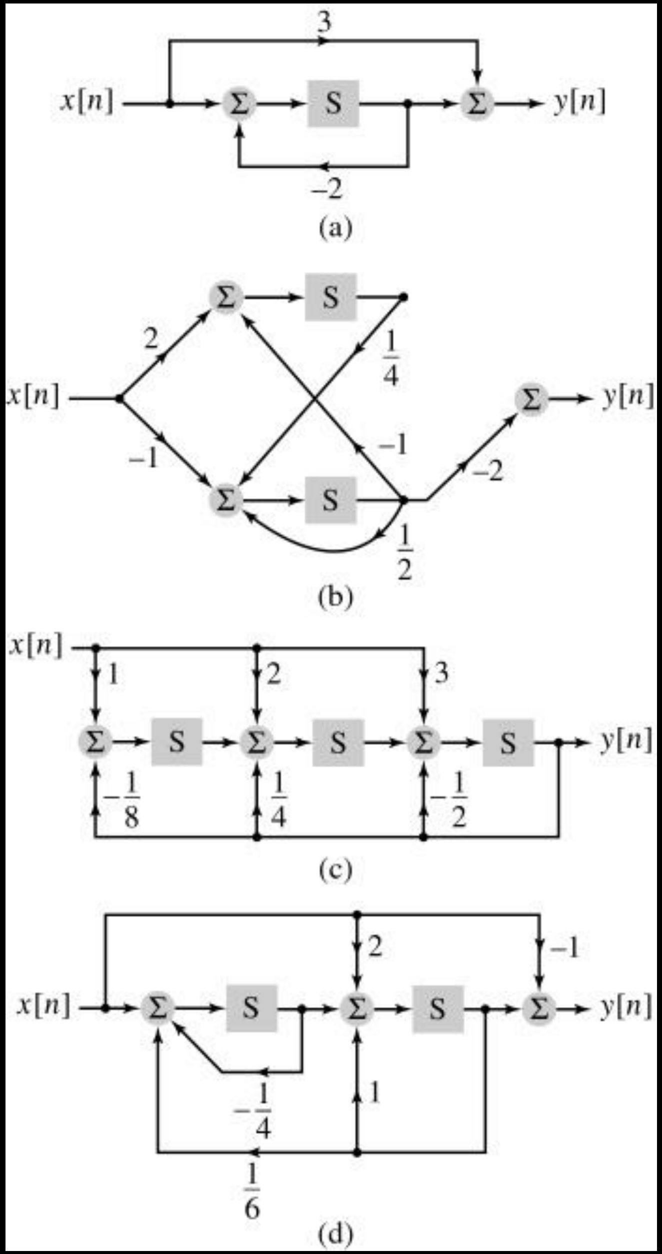


Figure P2.71 (p. 191)

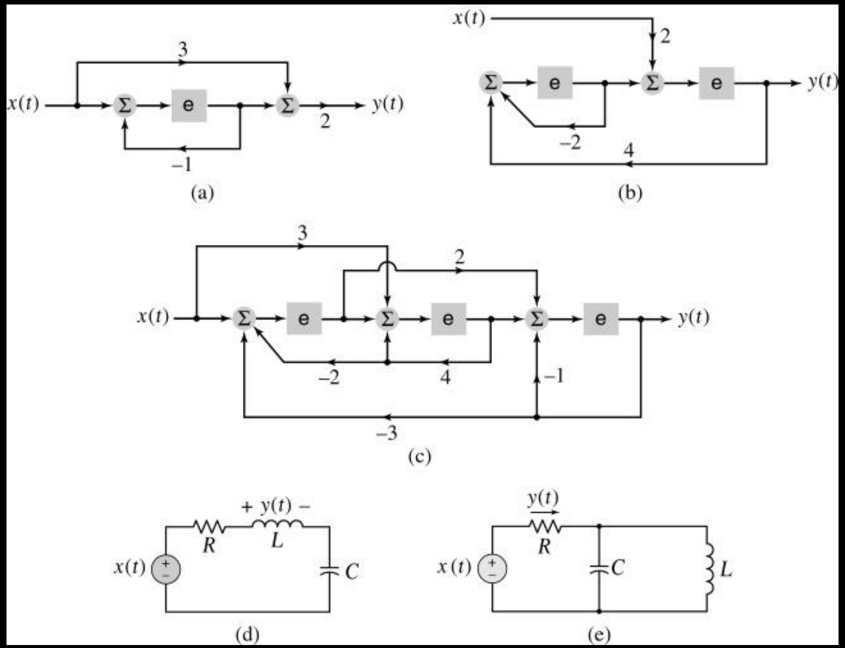


Figure P2.74 (p. 192)

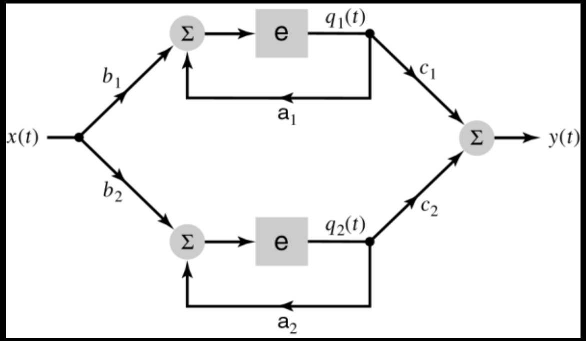


Figure P2.75 (p. 192)

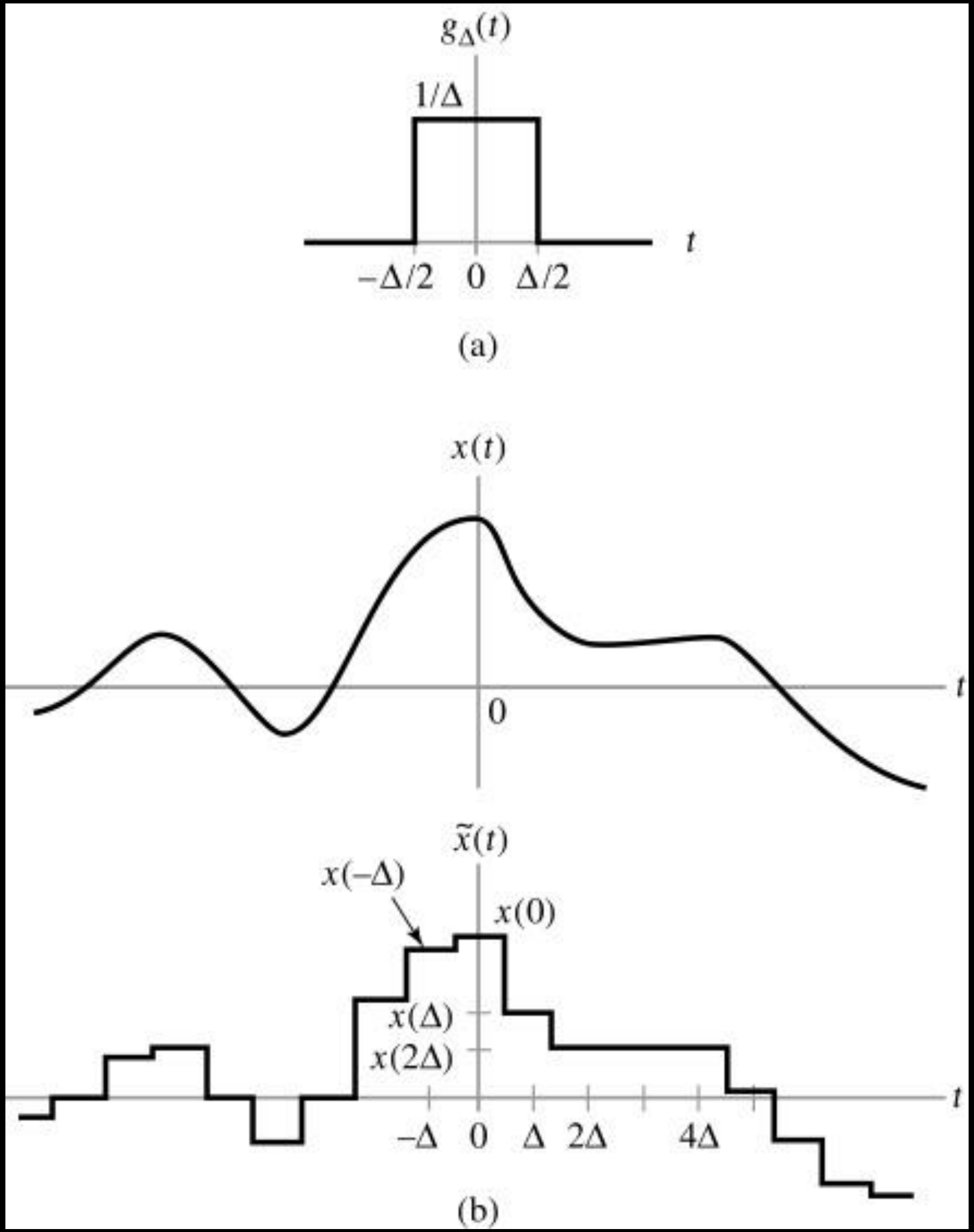


Figure P2.81 (p. 194)