Tutorial 1

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Fundamental Period of Continuous-Time and Discrete-Time Signals

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Fundamental Frequency / Fundamental Period of Continuous Time Signals

To identify the period T, the frequency $f = \frac{1}{T}$ or the angular frequency

$$w = 2\pi f = 2\pi/T$$

of a given sinusoidal or complex exponential signal, it is always helpful to write it in any of the following forms

$$\sin(wt) = \sin(2\pi f t) = \sin(2\pi t/T)$$

The fundamental frequency of a signal is the Greatest Common Divisor (GCD) of all the frequency components contained in a signal and equivalently, the fundamental period is the Least Common Multiple (LCM) of all individual periods of the components.

Example 1.a:
Fundamental Frequency of Continuous Time Signals

Find the fundamental frequency of the following continuous time signal

$$x(t) = cos\left(\frac{10\pi}{3}t\right) + sin\left(\frac{5\pi}{4}t\right)$$

The frequencies and periods of the two terms are, respectively,

$$w_1 = \frac{10\pi}{3}, f_1 = \frac{5}{3}, T_1 = \frac{3}{5} \text{ and } w_2 = \frac{5\pi}{4}, f_2 = \frac{5}{8}, T_2 = \frac{8}{5}$$

The fundamental frequency f_0 is the GCD of $f_1 = 5/3$ and $f_2 = 5/8$

$$f_0 = GCD\left(\frac{5}{3}, \frac{5}{8}\right) = GCD\left(\frac{40}{24}, \frac{15}{24}\right) = \frac{5}{24}$$

Example 1.b:
Fundamental Period of Continuous Time Signals

Alternatively, the period of the fundamental T_0 is the LCM of $T_1 = \frac{3}{5}$ and $T_1 = \frac{8}{5}$

$$T_0 = LCM\left(\frac{3}{5}, \frac{8}{5}\right) = \frac{24}{5}$$

Now we get $w_0 = 2\pi f_0 = \frac{2\pi}{T_0} = \frac{5\pi}{12}$ and the signal can be written as

$$x(t) = cos\left(8\frac{5\pi}{12}t\right) + sin\left(3\frac{5\pi}{12}t\right) = cos(8w_0t) + sin(3w_0t)$$

i.e., the two terms are the 3rd and 8th harmonic of the fundamental frequency w_0, respectively. See for online calculator https://www.calculator.net/lcm-calculator.html

Example 2.a: Find the fundamental frequency for the signal given

$$x(t) = \sin\left(\frac{5\pi}{6}t\right) + \cos\left(\frac{3\pi}{4}t\right) + \sin\left(\frac{\pi}{3}t\right)$$

The frequencies and periods of the three terms are, respectively,

$$w_1 = \frac{5\pi}{6}, f_1 = \frac{5}{12}, T_1 = \frac{12}{5}, w_2 = \frac{3\pi}{4}, f_2 = \frac{3}{8}, T_2 = \frac{8}{3}, w_3 = \frac{\pi}{3}, f_3 = \frac{1}{6}, T_3 = 6$$

The fundamental frequency f_0 is the GCD of f_1, f_2 and f_3:

$$f_0 = GCD\left(\frac{5}{12}, \frac{3}{8}, \frac{1}{6}\right) = GCD\left(\frac{10}{24}, \frac{9}{24}, \frac{4}{24}\right)$$

Example 2.a: Find the fundamental period for the signal given

Alternatively, the period of the fundamental T_0 is the LCM of T_1, T_2 and T_3:

$$T_0 = LCM\left(\frac{12}{5}, \frac{8}{3}, 6\right) = LCM\left(\frac{36}{15}, \frac{40}{15}, \frac{90}{15}\right)$$

The signal can be written as $x(t) = \sin\left(\frac{10\pi}{12}t\right) + \cos\left(\frac{9\pi}{12}t\right) + \sin\left(\frac{4\pi}{12}t\right)$

i.e., the fundamental frequency is $\omega_0 = \pi/12$, the fundamental period is $T = \frac{2\pi}{\omega_0} = 24$ and the three terms are the 4th, 9th and 10th harmonic of ω_0, respectively.

Example 3: Find the fundamental frequency of the following continuous time signal

$$x(t) = \cos\left(\frac{10}{3}t\right) + \sin\left(\frac{5\pi}{4}t\right)$$

Here the angular frequencies of the two terms are, respectively,

$$w_1 = \frac{10}{3}, \quad w_2 = \frac{5\pi}{4}$$

The fundamental frequency ω_0 should be the GCD of ω_1 and ω_2

$$w_0 = GCD\left(\frac{10}{3}, \frac{5\pi}{4}\right)$$

which does not exist since π is an irrational number and cannot be expressed as a ratio of two integers. Therefore, the two frequencies cannot be multiples of the same fundamental frequency. In other words, the signal as the sum of the two terms is not a periodic signal.

Note: The fundamental period T_0 of the periodic signal $x(t)$ is the smallest positive value of T for which the equation

$$x(t) = x(t + T)$$

holds.

The fundamental period T_0 must not necessarily be an integer, but it must be the smallest positive number corresponding to the smallest nonzero integer k such that $T_0 = \frac{2\pi k}{w_0}$.

Fundamental Period of Discrete Time Signals

For a discrete complex exponential $x[n] = e^{j\omega_1 n}$ to be periodic with period N, it has to satisfy

$$e^{jw_1(n+N)} = e^{jw_1 n} \text{i.e.,} \quad e^{jw_1 N} = e^{j2\pi k} = 1$$

that is, $\omega_1 N$ has to be multiple of 2π:

$$\Omega_1 N = 2\pi k \quad \text{i.e.,} \quad \frac{\Omega_1}{2\pi} = \frac{m}{N}$$

As m is an integer, $\Omega_1/2\pi$ must be a rational number (a ratio of two integers) so that the period $N = m2\pi/\Omega_1$ be the fundamental period, m be the smallest integer that makes N an integer, and the fundamental angular frequency is then

$$\Omega_0 = 2\pi/N = \Omega_1/m$$

The original signal can now be written as:

$$x[n] = e^{j\Omega_1 n} = e^{jm\Omega_0 n} = e^{jm2\pi n/N}$$

Example 1: Fundamental Frequency of Discrete Time Signals

Considering a discrete-time signal

$$x[n] = cos\left(\frac{\pi}{2}n\right) cos\left(\frac{\pi}{4}n\right), \quad n \in \mathrm{Z},$$

Period of the signal $x[n]$ can be found empirically as $2\pi/\frac{\pi}{4} = 8$ since the smaller sub-period is $\pi/4$. However, since the mathematical method give us more precise result, we shall refer to the trigonometric identities.

$$\cos(a)\cos(b) = \frac{1}{2}[\cos(a-b) + \cos\,(a+b)]$$

Hence, the waveform adopts the period of the lowest frequency because

$$\frac{\pi}{2}n - \frac{\pi}{4}n = \frac{\pi}{4}n \Rightarrow \frac{\pi}{4}n = \frac{2\pi}{8}n = \frac{2\pi}{N}n$$

Hence the period is $N = 8$.

Example 2: Find the fundamental period of $x[n] = sin\left(\frac{5\pi}{6}n\right) + cos\left(\frac{3\pi}{4}n\right)$

The Least-Common-Multiplier of the denominator is 12. Therefore

$$x[n] = sin\left(\frac{10\pi}{12}n\right) + cos\left(\frac{9\pi}{12}n\right)$$

Hence, the fundamental frequency is $w_0 = \frac{\pi}{12}$, the fundamental period is $T = \frac{2\pi}{w_0} = 24$ and the two terms are the 9th and 10th harmonic of the fundamental frequency w_0.

Example 3: Determine the fundamental period of the periodic signal

$$x[n] = 10cos\left(\frac{4\pi}{31}n\right)$$

We can rewrite the expression as

$$x[n] = 10cos\left(\frac{2\pi}{31/2}n\right)$$

Since N should be an integer, nearest multiple of 31/2, which are 31, 62 ... We take the nearest 31.

Periodicity of Discrete Time Signals

Consider next the discrete-time version of a sinusoidal signal, written as

$$x[n] = Acos(\Omega n + \phi) \qquad (1.39)$$

This discrete-time signal may or may not be periodic. For it to be periodic with a period of, say, N samples, it must satisfy Eq. (1.10) for all integer n and some integer N. Substituting $n + N$ for n in Eq. (1.39) yields

$$x[n + N] = Acos(\Omega n + \Omega \mathrm{N} + \phi)$$

To satisfy (1.10) we require that $\Omega N = 2\pi k$ radians or

$$\Omega = \frac{2\pi k}{N} \text{ radians/cycle for integer } k, N \qquad (1.40)$$

The important point to note here is that, unlike continuous-time sinusoidal signals, not all discrete-time sinusoidal systems with arbitrary values of Ω are periodic.

Specifically, for the discrete-time sinusoidal signal described in Eq. (1.39) to be periodic, the angular frequency Ω must be a rational multiple of 2π, as indicated in Eq. (1.40)

$$x[n] = x[n + N] \text{ for integer } n, \tag{1.10}$$

Example 1: **Discrete time sinusoidal signals**. A pair of sinusoidal signals with a common angular frequency is defined by $x_1[n] = \sin[5\pi n]$ and $x_2[n] = \sqrt{3}\cos[5\pi n]$.

a) Both $x_1[n]$ and $x_2[n]$ are periodic. Find the common fundamental period.

b) Express the composite sinusoidal signal $y[n] = x_1[n] + x_2[n]$ in the form of

$$y[n] = A cos(\Omega n + \phi)$$

and evaluate the amplitude A and phase ϕ.

Soln. 1.a) The angular frequency for both $x_1[n]$ and $x_2[n]$ is $\Omega = 5\pi$ radians/cycle.

Using
$$N = \frac{2\pi m}{\Omega} = \frac{2\pi m}{5\pi} = \frac{2m}{5} \qquad (1.40)$$

For $x_1[n]$ and $x_2[n]$ to be periodic, N must be an integer. This can be so only for

m = 5, 10, 15, … which results in N = 2, 4, 6, …

Example 1: **Discrete time sinusoidal signals**. A pair of sinusoidal signals with a common angular frequency is defined by $x_1[n] = \sin[5\pi n]$ and $x_2[n] = \sqrt{3}\cos[5\pi n]$.

a) Both $x_1[n]$ and $x_2[n]$ are periodic. Find the common fundamental period.

b) Express the composite sinusoidal signal $y[n] = x_1[n] + x_2[n]$ in the form of

$$y[n] = Acos(\Omega n + \phi)$$

Soln. 1.b) Recalling the trigonometric identity $Acos(\Omega \mathrm{n})cos(\phi) - Asin(\Omega \mathrm{n})sin(\phi)$ and letting $\Omega = 5\pi$, the RHS of the identity will resemble $x_1[n] + x_2[n]$. We can then write

$Asin(\phi) = -1$ and $Acos(\phi) = \sqrt{3}$. Hence, $\tan(\phi) = \frac{\sin(\phi)}{\cos(\phi)} = \frac{amplitude\ of\, x_1[n]}{amplitude\ of\ x_2[n]} = -\frac{1}{\sqrt{3}}$. From here, we find $\phi = -\pi/3$ radians. Substituting for this inequation $Asin(\phi) = -1$ and solving for the amplitude A, we get $A = -\frac{1}{\sin\left(-\frac{\pi}{2}\right)} = 2$. Accordingly, we find $y[n] = 2cos(5\pi n - \pi/3)$.

Example 2: Is the signal $x[n] = \sin(2n)$ periodic signal?

Solution: We can rewrite the signal $x[n] = \sin(2n)$ in the form

$$x[n] = \sin\left(\frac{2\pi n}{\pi}\right) = \sin\left(\frac{2\pi n}{N}\right).$$

There is no k that satisfies $2n = 2\pi k$. Hence the signal is NOT periodic

Example 3: Find the fundamental frequency and period of the signal

$$x[n] = \sin\left(\frac{5\pi}{6}n\right) + \cos\left(\frac{3\pi}{4}n\right) + \sin\left(\frac{\pi}{3}n\right)$$

$$x[n] = \sin\left(2\pi\frac{5}{12}n\right) + \cos\left(2\pi\frac{3}{8}n\right) + \sin\left(2\pi\frac{1}{6}n\right)$$

The Greatest Common Divisor (GCD) of the denominators is 12, therefore

$$x[n] = \sin\left(\frac{10\pi}{12}n\right) + \cos\left(\frac{9\pi}{12}n\right) + \sin\left(\frac{4\pi}{12}n\right)$$

i.e., the fundamental frequency is $\omega_0 = \pi/12$, the fundamental period is $T = \frac{2\pi}{\omega_0} = 24$ and the three terms are the 4th, 9th and 10th harmonic of ω_0, respectively.

Example 3: Use GCD to find the fundamental frequency and period of the signal

$$x[n] = \sin\left(\frac{5\pi}{6}n\right) + \cos\left(\frac{3\pi}{4}n\right) + \sin\left(\frac{\pi}{3}n\right)$$

Soln. Rewriting the equation as

$$x[n] = \sin\left(2\pi\frac{5}{12}n\right) + \cos\left(2\pi\frac{3}{8}n\right) + \sin\left(2\pi\frac{1}{6}n\right)$$

$$GCD\left(\frac{5}{12},\frac{3}{8},\frac{1}{6}\right) = GCD\left(\frac{10}{24},\frac{9}{24},\frac{4}{24}\right) = \frac{1}{24}(10,9,4), \qquad f_0 = \frac{1}{24}$$

$$x[n] = \sin\left(2\pi \times 10 \times \frac{1}{24}n\right) + \cos\left(2\pi \times 9 \times \frac{1}{24}n\right) + \sin\left(2\pi \times 4 \times \frac{1}{24}n\right)$$

10^{th} harmonic $\qquad$ 9^{th} harmonic $\qquad$ 4^{th} harmonic

Another method of solution uses Least Common Multiple (LCM) of discrete periods N_1, N_2, N_3 such as

$$x[n] = \sin\left(2\pi \frac{5}{12} n\right) + \cos\left(2\pi \frac{3}{8} n\right) + \sin\left(2\pi \frac{1}{6} n\right)$$

$$\text{LCM}\left(\frac{12}{5}, \frac{8}{3}, \frac{6}{1}\right) = \text{LCM}\left(\frac{36}{15}, \frac{40}{15}, \frac{90}{15}\right) = \frac{360}{15} = 24 = T_0$$