- **Strict and Wide Sense Stationarity**
- **Autocorrelation Function of a Stationary Process**
- **Power Spectral Density**

Stationary Random Processes

- Stationarity refers to *time invariance* of some, or all, the statistics of a random process, *e.g.*, mean, autocorrelation, nth order distribution, *etc*
- We define two types of stationarity, *strict sense* (SSS) and *wide sense* (WSS)
- A random process $X(t)$ (or X_n) is said to be SSS if *all* its finite order distributions are time invariant, *i.e.*, the joint cdf (pdf, or pmf) of $X(t_1), X(t_2), \ldots, X(t_k)$ is the same as for $X(t_1+\alpha), X(t_2+\alpha)$, $\ldots, X(t_k+\alpha)$, for all k, all $t_1, t_2, \ldots, t_k$, and all time shifts α
- So for a SSS process, the first order distribution is independent of t, and the second order distribution, *i.e.*, the distribution of any two samples $X(t_1)$ and $X(t_2)$, depends only on $\tau = t_2 - t_1$

  To see this, note that from the definition of stationarity, for any t, the joint distribution of $X(t_1)$ and $X(t_2)$ is the same as the joint distribution of $X(t_1 + (t - t_1))$ and $X(t_2 + (t - t_1)) = X(t + (t_2 - t_1))$

Wide Sense Stationary Random Processes

- A random process $X(t)$ is said to be WSS if its mean and autocorrelation functions are time invariant, *i.e.*, $E(X(t)) = \mu$, independent of t and $R_X(t_1, t_2)$ is only a function of $(t_2 - t_1)$
- Since $R_X(t_1, t_2) = R_X(t_2, t_1)$, if $X(t)$ is WSS, $R_X(t_1, t_2)$ is only a function of $|t_2 - t_1|$
- Clearly SSS $\Rightarrow$ WSS, the converse, however, is not necessarily true

- For GRP, WSS $\Rightarrow$ SSS, since the process is completely specified by its mean and autocorrelation functions
- Random walk is not WSS, since $R_X(n_1, n_2) = \min\{n_1, n_2\}$ is not time invariant – in fact no independent increment process can be WSS

Autocorrelation Function of WSS Processes

- Let $X(t)$ be a WSS process and relabel $R_X(t_1, t_2)$ as $R_X(\tau)$, where $\tau = t_2 - t_1$

1. $R_X(\tau)$ is real and even, *i.e.*, $R_X(\tau) = R_X(-\tau)$ for all τ
2. $|R_X(\tau)| \leq R_X(0) = E(X^2(t))$, the "average power" of $X(t)$

   This can be shown using the Schwartz inequality

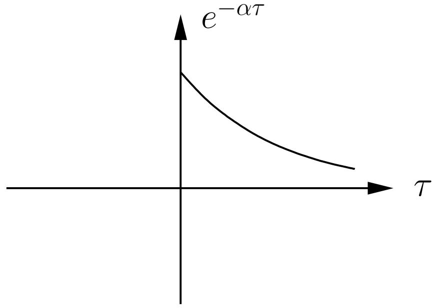
   For any t

$$
\begin{aligned}
(R_X(\tau))^2 &= (E(X(t)X(t+\tau)))^2 \\
&\leq E(X^2(t))E(X^2(t+\tau)) = (R_X(0))^2
\end{aligned}
$$

3. If $R_X(T) = R_X(0)$ for some T, then $R_X(\tau)$ is periodic with period T and so is $X(t)$ (with probability 1)

Which Functions can be an $R_X(\tau)$?

1.



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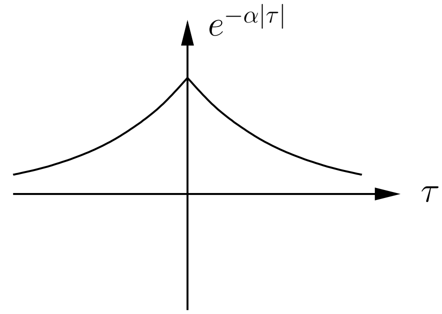

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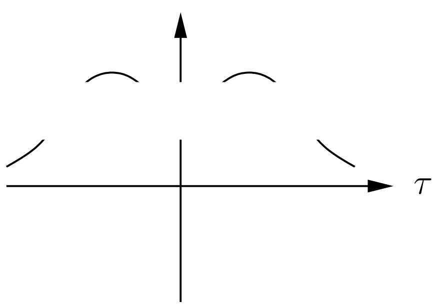

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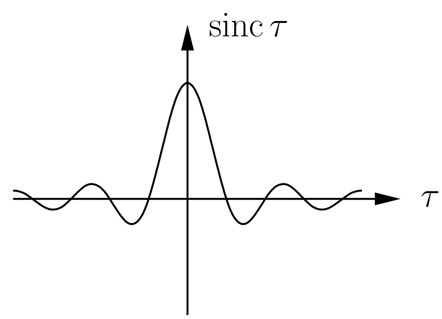

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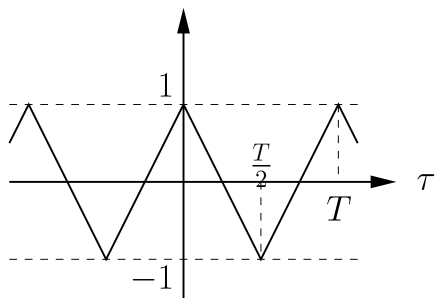

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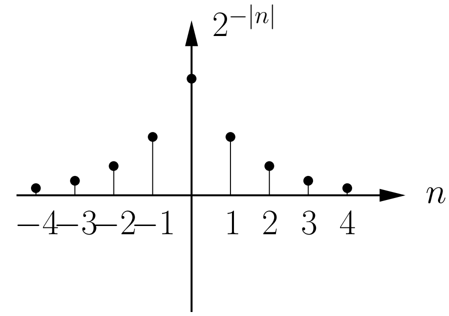

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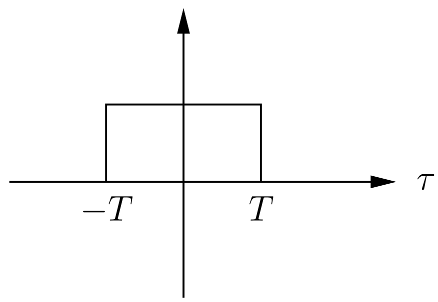
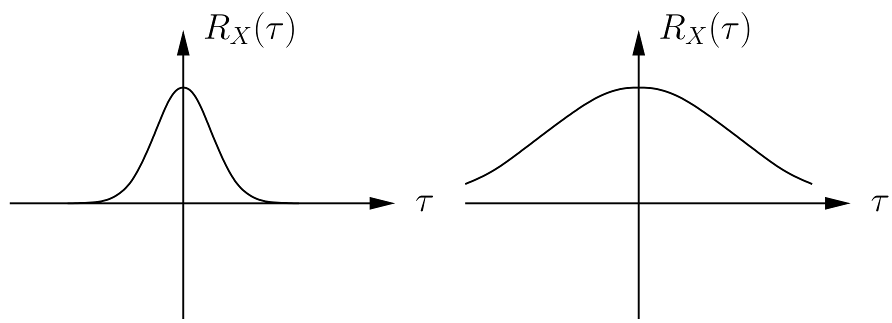


Interpretation of Autocorrelation Function

- If $R_X(\tau)$ drops quickly with τ, this means that samples become uncorrelated quickly as we increase τ, conversely, if $R_X(\tau)$ drops slowly with τ, samples are highly correlated



- So $R_X(\tau)$ is a measure of the rate of change of $X(t)$ with time t, $i.e.$, "the frequency response of $X(t)$"
- It turned out that this is not just an interpretation – the Fourier Transform of $R_X(\tau)$ (the power spectral density) is in fact the average power density over frequency

Power Spectral Density

- The *power spectral density* (psd) of a WSS random process $X(t)$, is the Fourier Transform of $R_X(\tau)$,

$$S_X(f) = \mathcal{F}[R_X(\tau)] = \int_{-\infty}^{\infty} R_X(\tau) e^{-j2\pi\tau f} d\tau$$

- For a discrete time process X_n, the power spectral density is the discrete Fourier Transform (DFT) of the sequence $R_X(n)$,

$$S_X(f) = \sum_{n=-\infty}^{\infty} R_X(n) e^{-j2\pi n f}, \text{ for } |f| < \frac{1}{2}$$

- $R_X(\tau)$ (or $R_X(n)$) can be recovered from $S_X(f)$ by taking the inverse Fourier Transform, *i.e.*,

$$R_X(\tau) = \int_{-\infty}^{\infty} S_X(f) e^{j2\pi\tau f} df, \text{ and inverse DFT,}$$

$$R_X(n) = \int_{-\frac{1}{2}}^{\frac{1}{2}} S_X(f) e^{j2\pi n f} df$$

Properties of the Power Spectral Density

1. $S_X(f)$ is real and even, since the Fourier Transform of a real and even function is real and even ($R_X(\tau)$ is real and even)
2. $S_X(f)$ is the power density, *i.e.*, the average power of $X(t)$ in frequency band $[f_1, f_2]$ is $2\int_{f_1}^{f_2} S_X(f) df$

- From (2) it follows that

$$S_X(f) \geq 0, \text{ and}$$

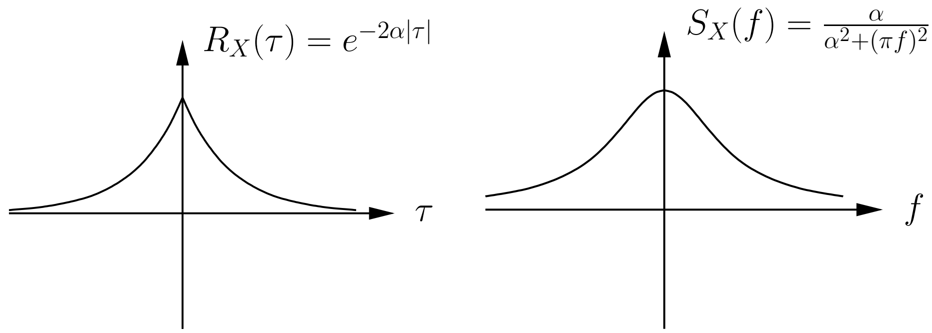
$$E(X^2(t)) = \int_{-\infty}^{\infty} S_X(f) df,$$

i.e., the average power of $X(t)$ is the area under $S_X(f)$

- In general a function $S(f)$ is a psd iff it is real, even, nonnegative, and $\int_{-\infty}^{\infty} S(f) df < \infty$

Examples

1.



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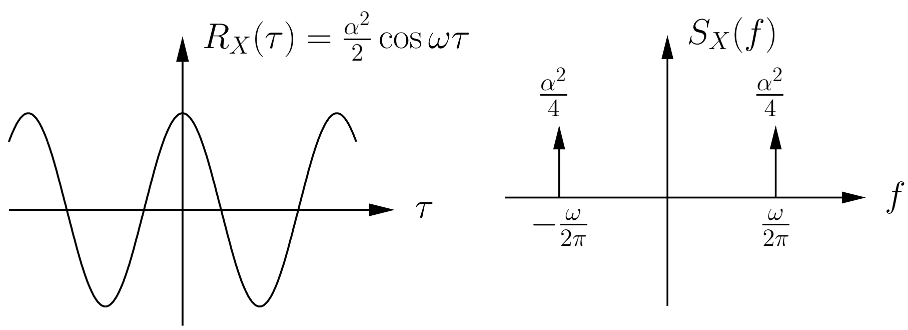

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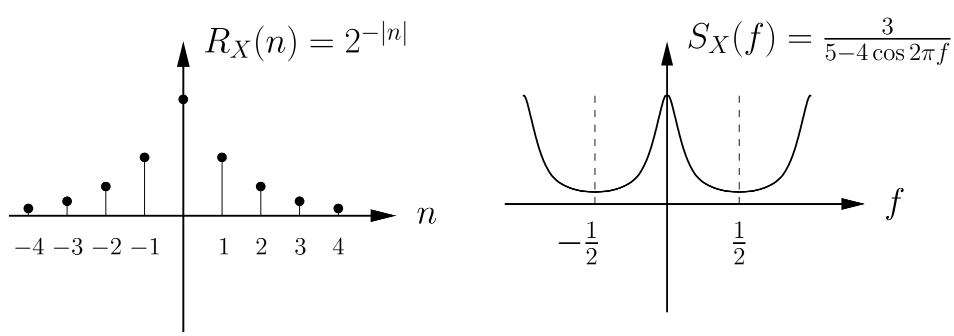
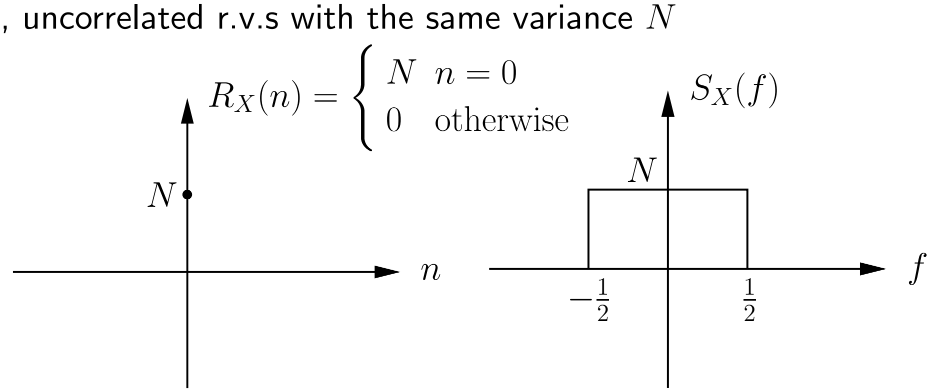
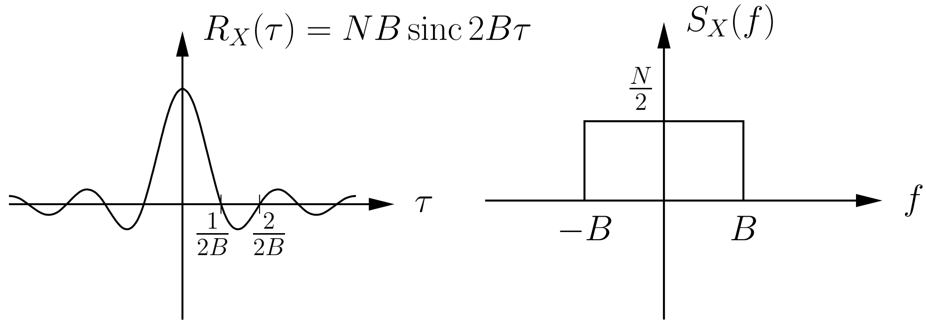


4. Discrete time white noise process: X_n such that $X_1, X_2, \ldots$ are zero mean, uncorrelated r.v.s with the same variance N



If X_n is also a GRP, then we get a discrete time WGN process

5. Bandlimited white noise process: WSS zero mean process $X(t)$ with

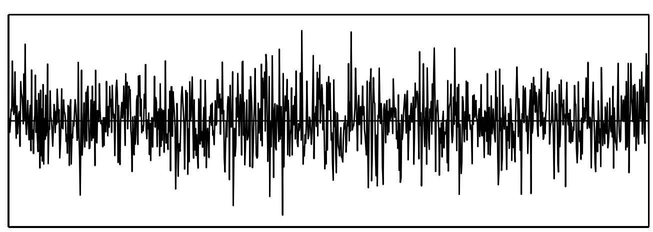


For any t, the samples $X(t \pm \frac{n}{2B})$, for $n = 0, 1, 2, \ldots$, are uncorrelated

6. White noise process: Now if we let $B \to \infty$ in the previous example, we get a *white noise process*, which has

$$S_X(f) = \frac{N}{2}, \text{ for all } f, \text{ and}$$
$$R_X(\tau) = \frac{N}{2}\delta(\tau)$$

If, in addition, $X(t)$ is a GRP, then we get the famous white gaussian noise (WGN) process



- Remarks on white noise:
  - For a white noise process all samples are uncorrelated
  - The process is not physically realizable, since it has infinite power
  - However, it plays a similar role in random processes to the role of a point mass in physics and delta function in EE
  - Thermal and shot noise are well modelled as white gaussian noise, since they have very flat psd over very wide band (GHzs)