EENG226
Signals and Systems

Chapter 1.7-1.9
Introduction to Signals and Systems

Properties of Systems

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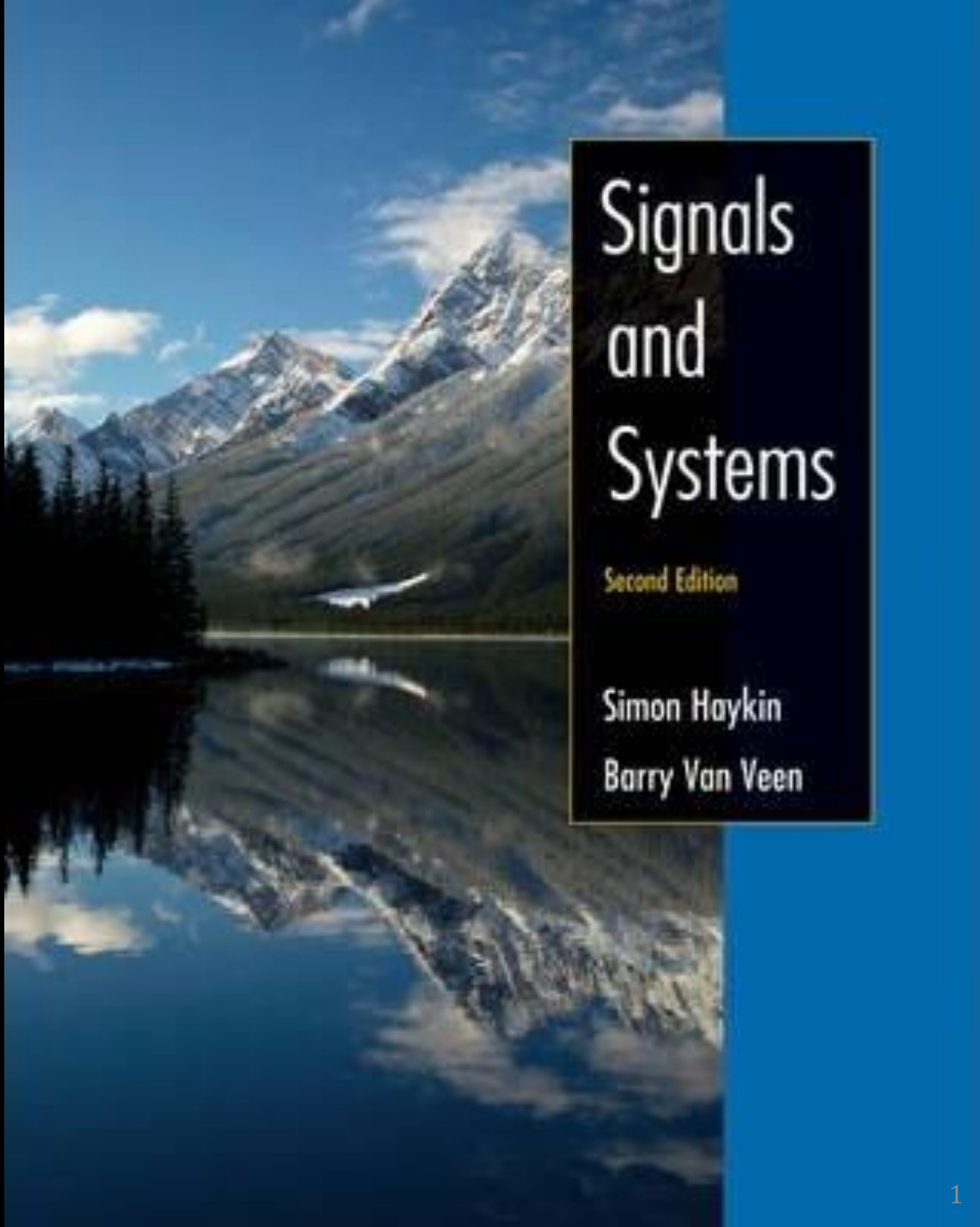
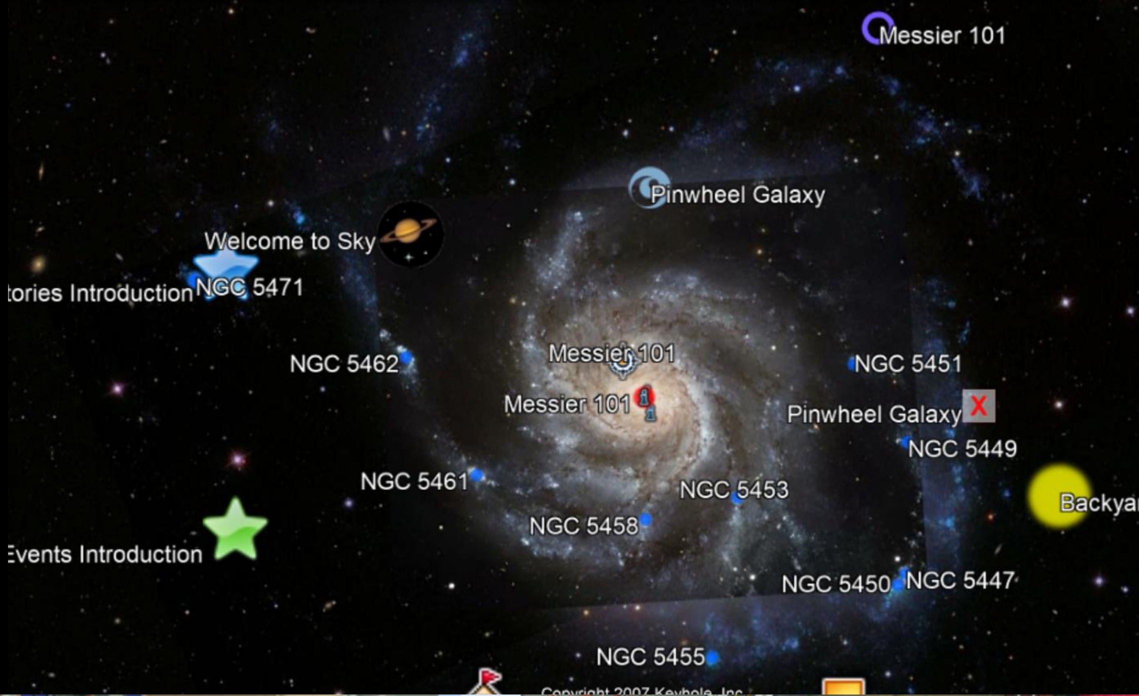
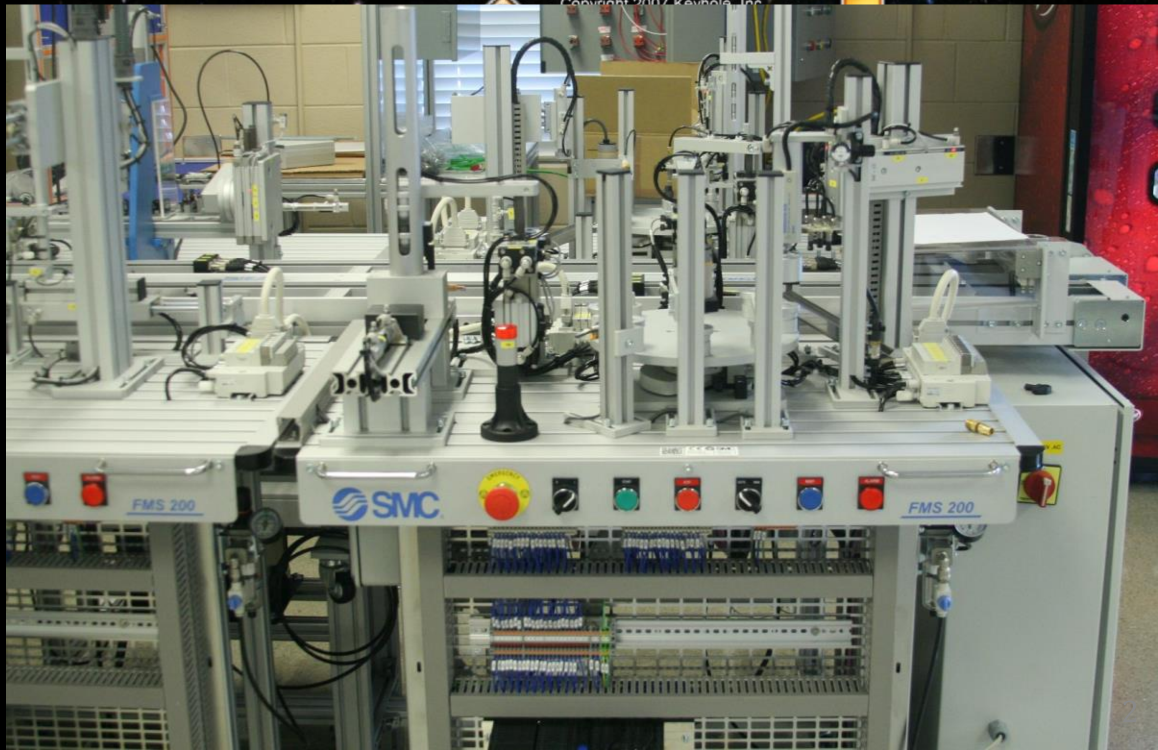


Chapter 1: Introduction

Objectives of this chapter





1.7 Systems Viewed as Interconnections of Operations

- A system may be viewed as an interconnection of operations that transforms an input signal into an output signal with properties different from the input signal
- Let the overall operator H denote the action of a system. Then the application of a continuous-time signal $x(t)$ to the input of the system yields the output signal

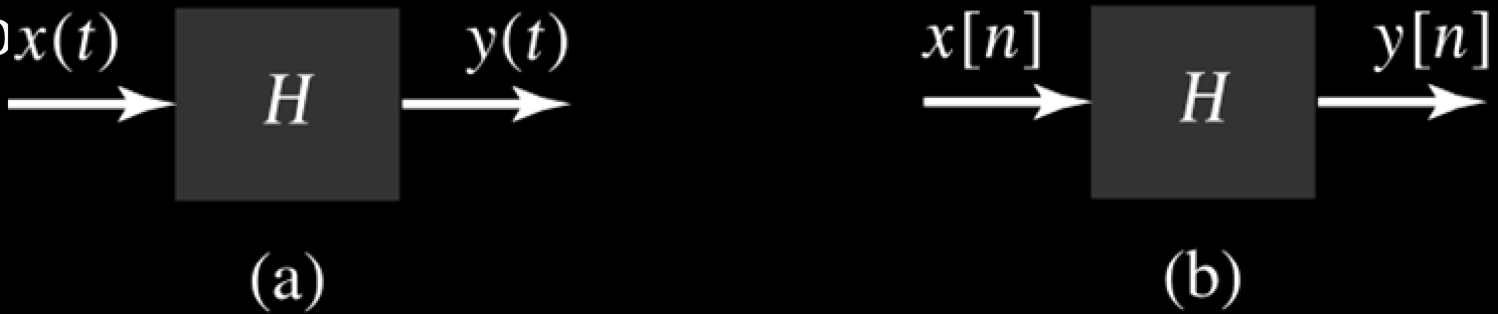
$$y(t) = \mathrm{H}\{x(t)\} \tag{1.78}$$

- the discrete-time case

$$y[n] = \mathrm{H}\{x(n)\} \tag{1.79}$$

- As in Fig. 1.49.

Figure 1.49 (p. 53)
Block diagram representation o
operator *H* for
(a) continuous time and
(b) discrete time.



(a)

(b)

EXAMPLE 1.12 MOVING-AVERAGE SYSTEM Consider a discrete-time system whose output signal $y[n]$ is the average of the three most recent values of the input signal $x[n]$; that is,

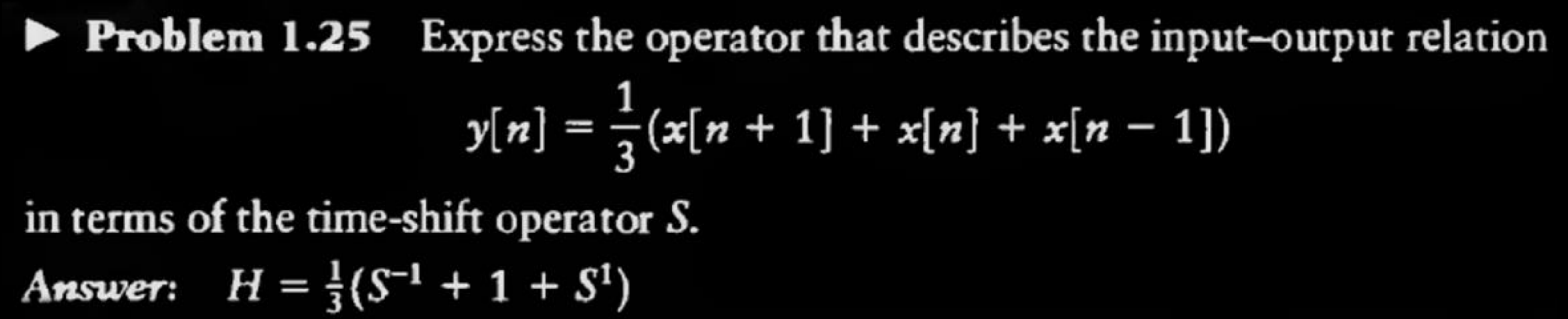
$$y[n] = \frac{1}{3}(x[n] + x[n-1] + x[n-2]).$$

Such a system is referred to as a *moving-average system*, for two reasons. First, $y[n]$ is the average of the sample values $x[n]$, $x[n-1]$, and $x[n-2]$. Second, the value of $y[n]$ changes as n moves along the discrete-time axis. Formulate the operator H for this system; hence, develop a block diagram representation for it.

Solution: Let the operator S^k denote a system that shifts the input $x[n]$ by k time units to produce an output equal to $x[n-k]$, as depicted in Fig. 1.50. Accordingly, we may define the overall operator H for the moving-average system as

$$H = \frac{1}{3}(1 + S + S^2)$$

Two different implementations of H (i.e., the moving-average system) that suggest themselves are presented in Fig. 1.51. The implementation shown in part (a) of the figure uses the *cascade* connection of two identical unity time shifters, namely, $S^1 = S$. By contrast, the implementation shown in part (b) of the figure uses two different time shifters, S and S^2, connected in *parallel*. In both cases, the moving-average system is made up of an interconnection of three functional blocks, namely, two time shifters and an adder, connected by a scalar multiplication. ■



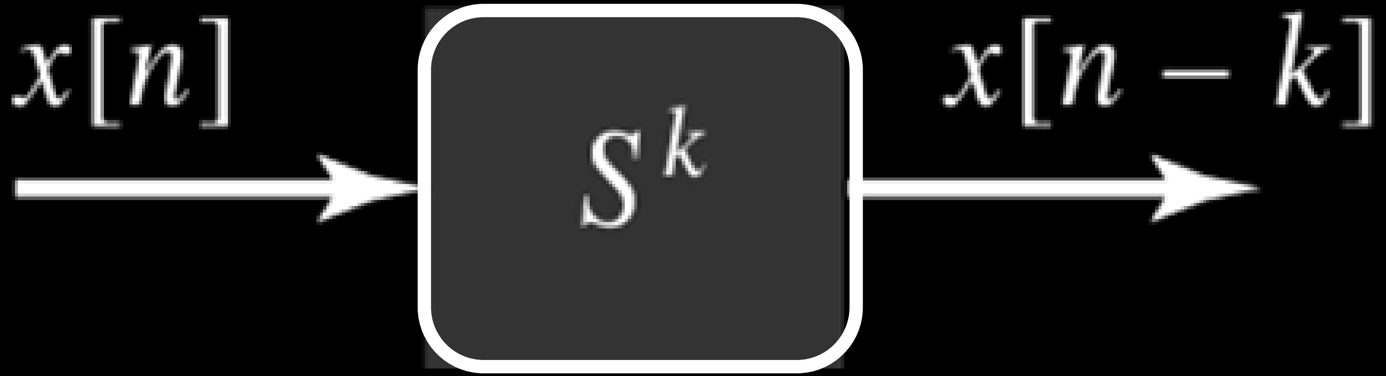
▶ **Problem 1.25** Express the operator that describes the input–output relation

$$y[n] = \frac{1}{3}(x[n+1] + x[n] + x[n-1])$$

in terms of the time-shift operator S.

Answer: $H = \frac{1}{3}(S^{-1} + 1 + S^{1})$

Figure 1.50 (p. 54)
Discrete-time-shift operator S^k, operating on the discrete-time signal $x[n]$ to produce $x[n - k]$.



In the interconnected systems shown in Figs. 1.51(a) and (b), the signal flows through each one of them in the forward direction only.

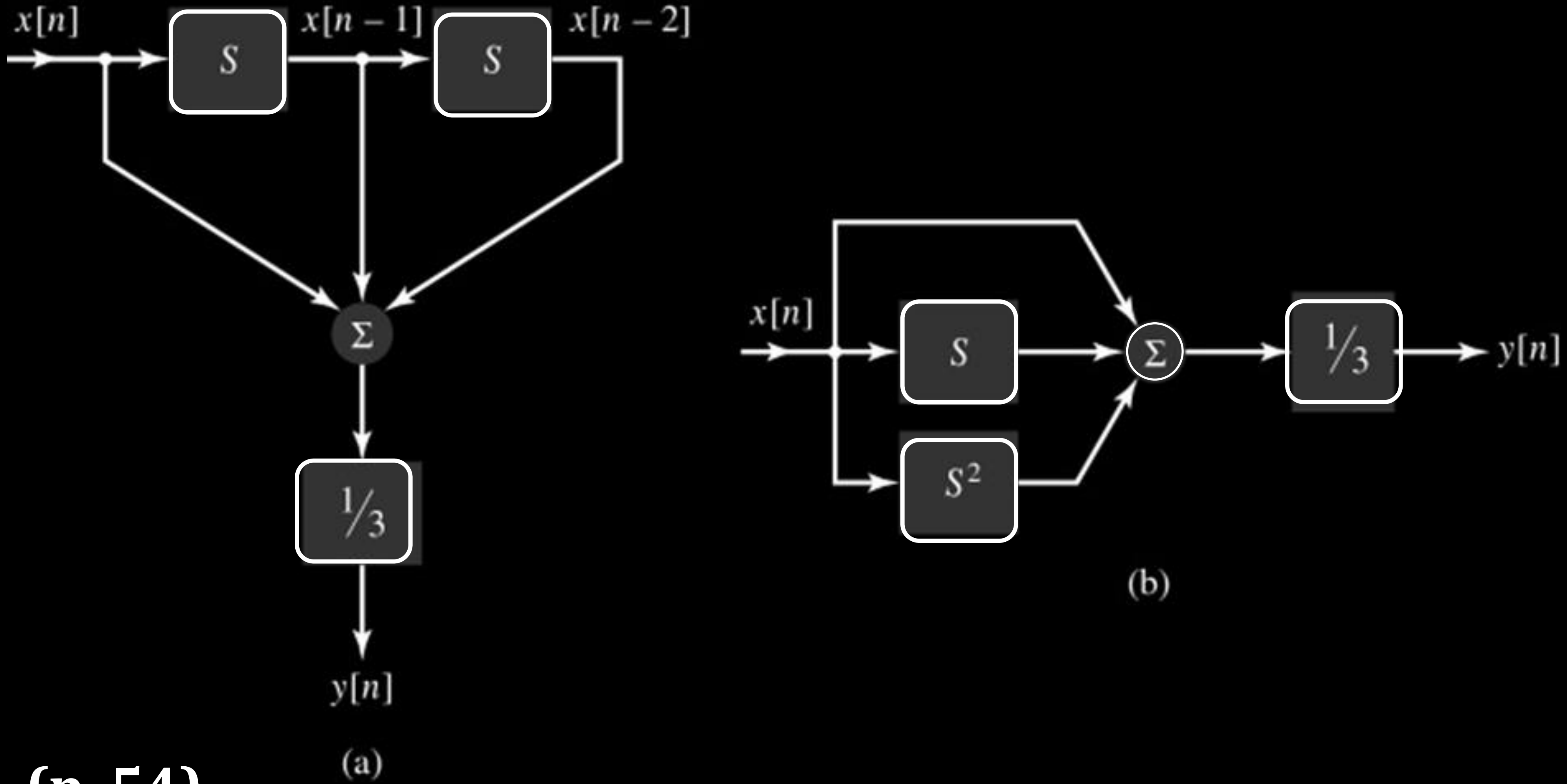


Figure 1.51 (p. 54)
Two different (but equivalent) implementations of the moving-average system: (a) cascade form of implementation and (b) parallel form of implementation.

1.8 Properties of Systems

1.8.1 Stability

- A system is said to be bounded-input, bounded-output (BIBO) stable iff every bounded input results in a bounded output
- Considering a continuous-time system whose input-output relation is as described in (1.78). Operator H is BIBO stable if output signal $y(t)$ satisfies

$$|y(t)| \leq M_y < \infty \text{ for all } t \qquad (1.80)$$

- whenever the input signals $x(t)$ satisfy the condition

$$x(\mathrm{t}) \leq \mathrm{M_x} < \infty \text{ for all t} \qquad (1.81)$$

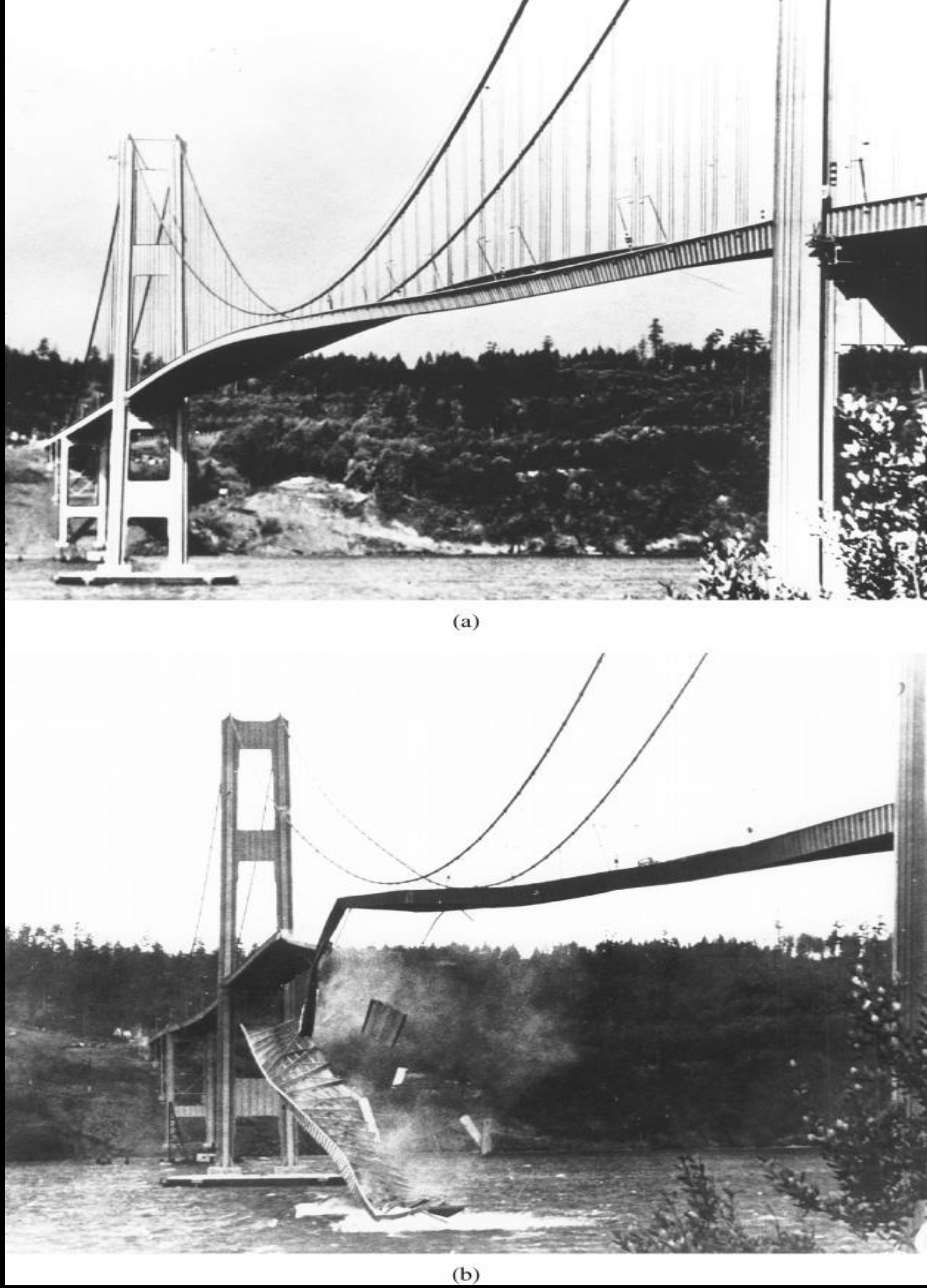
- From engineering perspective, system remain stable under all conditions.
- One famous example of unstable system is first Tacoma Narrows suspension bridge, which collapsed on 7 Nov. 1940, due to wind-induced vibrations.

Figure 1.52a (p. 56)
Dramatic photographs showing the collapse of the Tacoma Narrows suspension bridge on November 7, 1940.

(a) Photograph showing the twisting motion of the bridge’s center span just before failure.

(b) A few minutes after the first piece of concrete fell, this second photograph shows a 600-ft section of the bridge breaking out of the suspension span and turning upside down as it crashed in Puget Sound, Washington. Note the car in the top right-hand corner of the photograph.

(Courtesy of the Smithsonian Institution.)



EXAMPLE 1.13 MOVING-AVERAGE SYSTEM (CONTINUED) Show that the moving-average system described in Example 1.12 is BIBO stable.

Solution: Assume that

$$|x[n]| < M_x < \infty \quad \text{for all } n.$$

Using the given input–output relation

$$y[n] = \frac{1}{3}(x[n] + x[n-1] + x[n-2]),$$

we may write

$$\begin{aligned} |y[n]| &= \frac{1}{3}|x[n] + x[n-1] + x[n-2]| \\ &\le \frac{1}{3}(|x[n]| + |x[n-1]| + |x[n-2]|) \\ &\le \frac{1}{3}(M_x + M_x + M_x) \\ &= M_x. \end{aligned}$$

Hence, the absolute value of the output signal $y[n]$ is always less than the maximum absolute value of the input signal $x[n]$ for all n, which shows that the moving-average system is stable. ■

EXAMPLE 1.14 UNSTABLE SYSTEM Consider a discrete-time system whose input–output relation is defined by

$$y[n] = r^n x[n],$$

where $r > 1$. Show that this system is unstable.

Solution: Assume that the input signal $x[n]$ satisfies the condition

$$|x[n]| \le M_x < \infty \quad \text{for all } n.$$

We then find that

$$\begin{aligned} |y[n]| &= |r^n x[n]| \\ &= |r^n| \cdot |x[n]|. \end{aligned}$$

With $r > 1$, the multiplying factor r^n diverges for increasing n. Accordingly, the condition that the input signal is bounded is not sufficient to guarantee a bounded output signal, so the system is unstable. To prove stability, we need to establish that all bounded inputs produce a bounded output. ■

1.8.2 Memory

- A system is said to possess memory if its output signal depends on past or future values of the input signal
- In contrast, a system is said to be memoryless if its output signal depends only on the present value of the input signal
- A resistor is memoryless, since the current $i(t) = \frac{1}{R} v(t)$
- An inductor has memory since $i(t) = \frac{1}{L} \int_{-\infty}^{t} v(\tau) d\tau$
- Moving-average system of Example 1.12 described by the input-output relation

$$\mathrm{y[n]} = \frac{1}{3}(x[n] + x[n-1] + x[n-2])$$

- has memory, since value of output signal $y[n]$ at time n depends on the present and on two past values of the input signal $y[n]$
- $y[n] = x^2[n]$ is memoryless.

1.8.3 Causality

- A system is said to be causal if the present value of the output signal depends only on the present or past values of the input signal.
- In contrast, the output signal of a noncausal system depends on one or more future values of the input signal.
- For example, the moving-average system described by following eqn is causal

$$\mathrm{y[n]} = \frac{1}{3}(x[n] + x[n-1] + x[n-2])$$

- But the following system is noncausal

$$\mathrm{y[n]} = \frac{1}{3}(x[n+1] + x[n] + x[n-1])$$

Problem 1.30

Consider the RC circuit shown in Fig. 1.53 with input voltage $v_1(t)$ and output voltage $v_2(t)$. Is this system causal or noncausal?

Answer: Causal.

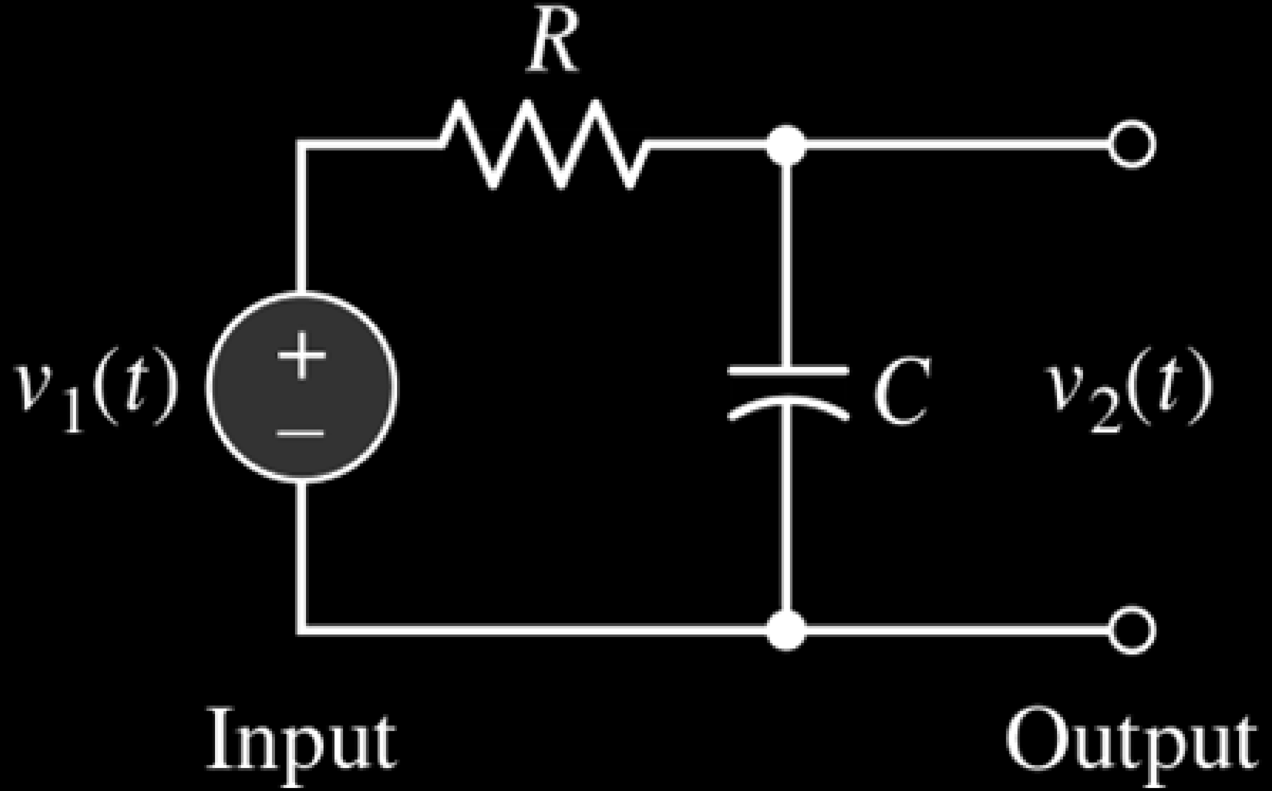


Figure 1.53 (p. 59)

Series *RC* circuit driven from an ideal voltage source $v_1(t)$, producing output voltage $v_2(t)$.

1.8.4 Invertibility

- A system is said to be invertible if the input of the system can be recovered from the output.

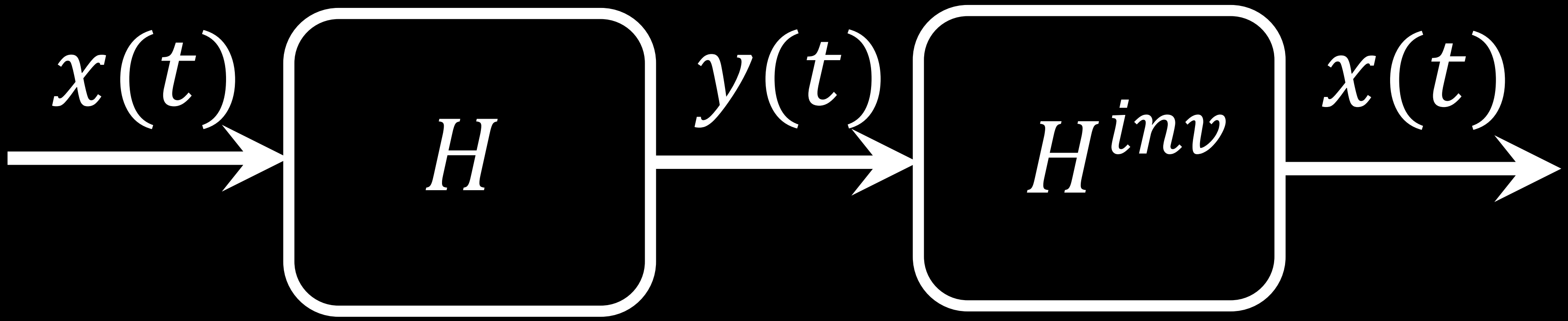
- Let the operator H represent a continuous-time system, with the input signal $x(t)$ producing the output signal $y(t)$. Let the output signal $y(t)$ be applied to a second continuous-time system represented by the operator H^{inv} as illustrated in Fig. 1.54. Then output signal of the second system is defined by

$$H^{inv}\{y(t)\} = H^{inv}\{H\{x(t)\}\} = H^{inv}H\{x(t)\}$$

$$H^{inv}H = I \qquad (1.82)$$

- Where $\boldsymbol{I}$ is the identity operator and H^{inv} is the inverse operator.

Figure 1.54 (p. 59)
The notion of system invertibility. The second operator H^{inv} is the inverse of the first operator H. Hence, the input $x(t)$ is passed through the cascade correction of H and H^{-1} completely unchanged.



EXAMPLE 1.15 INVERSE OF SYSTEM Consider the time-shift system described by the input–output relation

$$y(t) = x(t - t_0) = S^{t_0}\{x(t)\},$$

where the operator S^{t_0} represents a time shift of t_0 seconds. Find the inverse of this system.

Solution: For this example, the inverse of a time shift of t_0 seconds is a time shift of $-t_0$ seconds. We may represent the time shift of $-t_0$ by the operator S^{-t_0}, which is the inverse of S^{t_0}. Thus, applying S^{-t_0} to the output signal of the given time-shift system, we get

$$\begin{aligned} S^{-t_0}\{y(t)\} &= S^{-t_0}\{S^{t_0}\{x(t)\}\} \\ &= S^{-t_0}S^{t_0}\{x(t)\}. \end{aligned}$$

For this output signal to equal the original input signal $x(t)$, we require that

$$S^{-t_0}S^{t_0} = I,$$

which is in perfect accord with the condition for invertibility described in Eq. (1.82). ■

1.8.5 Time Invariance

- A system is said to be time invariant if a time delay or time advance of the input signal leads to an identical time shift in the output signal. This implies that a time-invariant system responds identically no matter when the input signal is applied.

- Consider a continuous-time system whose input-output relation is described by (1.78), in the form $y_1(t)$ = H{$x_1(t)$}

- Let $y_2(t)$ denote the output signal of the system H produced in response to the time-shifted input $x_1\{t$ - $t_0)$. We may then write

$$y_2(t) = H\{x_1(t\text{-}t_0)\}= H\{S^{t_0}x_1(t\text{-}t_0)\}= HS^{t_0}\{x_1(t\text{-}t_0)\}= S^{t_0}H\{x_1(t\text{-}t_0)\} \qquad (1.83)$$

- which is represented by the block diagram shown in Fig. 1.55(a) and where the operator S^{t_0} represents a time shift equal to t_0 seconds

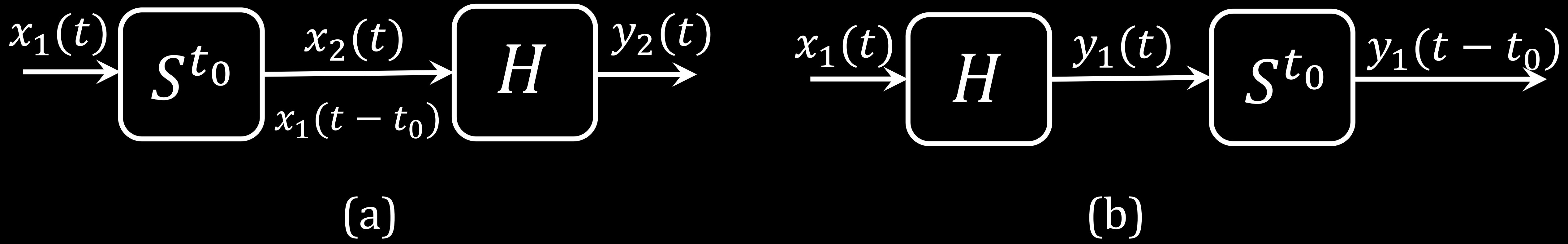


Figure 1.55 (p. 61): The notion of time invariance.
(a) Time-shift operator S^{t_0} preceding operator H.
(b) Time-shift operator S^{t_0} following operator H.

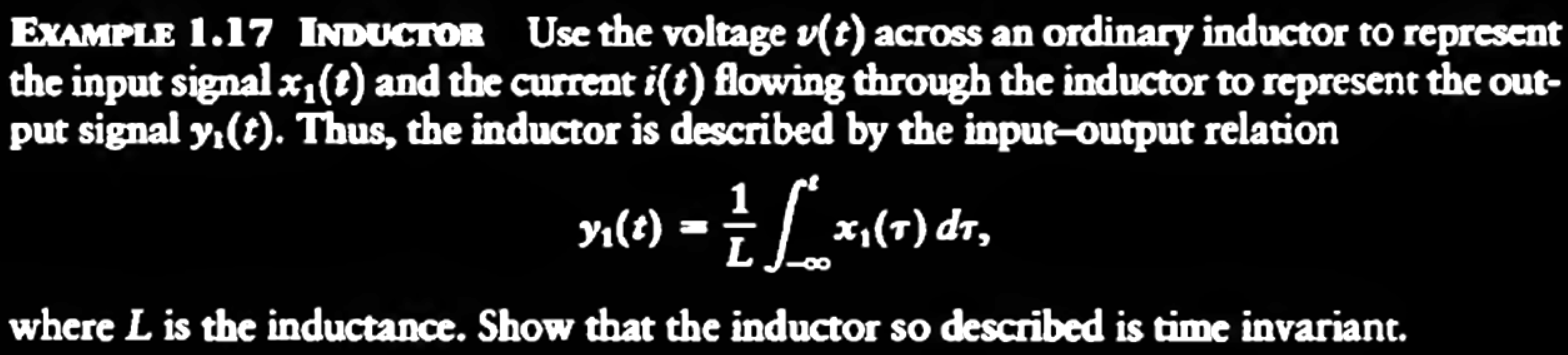
Note that these two situations are equivalent, provided that H is time invariant.

Problem 1.33

Is the following discrete-time system described by the input-output relation time invariant?

$$y[n] = r^n x[n]$$

Answer: No



Example 1.17 Inductor Use the voltage $v(t)$ across an ordinary inductor to represent the input signal $x_1(t)$ and the current $i(t)$ flowing through the inductor to represent the output signal $y_1(t)$. Thus, the inductor is described by the input–output relation

$$y_1(t) = \frac{1}{L}\int_{-\infty}^{t} x_1(\tau)\, d\tau,$$

where L is the inductance. Show that the inductor so described is time invariant.

PTO for solution…

Solution: Let the input $x_1(t)$ be shifted by t_0 seconds, yielding $x_1(t - t_0)$. The response $y_2(t)$ of the inductor to $x_1(t - t_0)$ is

$$y_2(t) = \frac{1}{L} \int_{-\infty}^{t} x_1(\tau - t_0)\, d\tau.$$

Next, let $y_1(t - t_0)$ denote the original output of the inductor, shifted by t_0 seconds; that is,

$$y_1(t - t_0) = \frac{1}{L} \int_{-\infty}^{t-t_0} x_1(\tau)\, d\tau.$$

Although at first examination $y_2(t)$ and $y_1(t - t_0)$ look different, they are in fact equal, as shown by a simple change in the variable of integration. Let

$$\tau' = \tau - t_0.$$

Then for a constant t_0, we have $d\tau' = d\tau$. Hence, changing the limits of integration, the expression for $y_2(t)$ may be rewritten as

$$y_2(t) = \frac{1}{L} \int_{-\infty}^{t-t_0} x_1(\tau')\, d\tau',$$

which, in mathematical terms, is identical to $y_1(t - t_0)$. It follows that an ordinary inductor is time invariant. ■

1.8.6 Linearity

- A system is said to be linear in terms of the system input (excitation) $x(t)$ and the system output (response) $y(t)$ if it satisfies the following two properties of superposition and homogeneity:

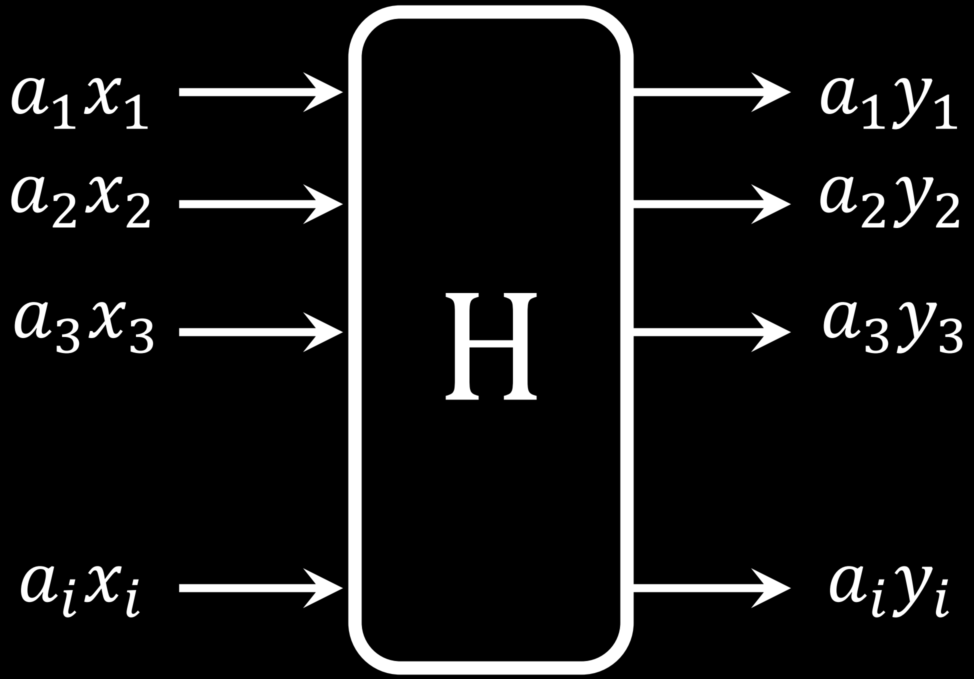
1. **Superposition:** If an input $x(t) = x_1(t)$ produces an output $y(t) = y_1(t)$ and an input $x(t) = x_2(t)$ produces an output $y(t) = y_2(t)$,

   Then for the system to be linear, it is necessary that $x(t) = x_1(t) + x_2(t)$ produce the corresponding output $y(t) = y_1(t) + y_2(t)$

2. **Homogeneity:** If an input $x(t)$ results in an output $y(t)$, then the system is said to exhibit the property of homogeneity if, whenever the input x(t) is scaled by a constant factor a, the output y(t) is scaled by exactly the same constant factor a.

1.8.6 Linearity

- If $x(t) = \sum_{i=1}^{N} a_i x_i(t)$ produces
- $y(t) = H\{x(t)\} = H\left\{\sum_{i=1}^{N} a_i x_i(t)\right\} = \sum_{i=1}^{N} a_i y_i(t)$
- then, the system is linear



Example 1.19 Linear Discrete-Time System

Consider a discrete-time system described by the input-output relation

$$y[n] = nx[n]$$

Show that this system is linear.

Solution: Let the input signal $x[n]$ be expressed as the weighted sum

$$x[n] = \sum_{i=1}^{N} a_i x_i[n]$$

We may then express the resulting output signal of the system as

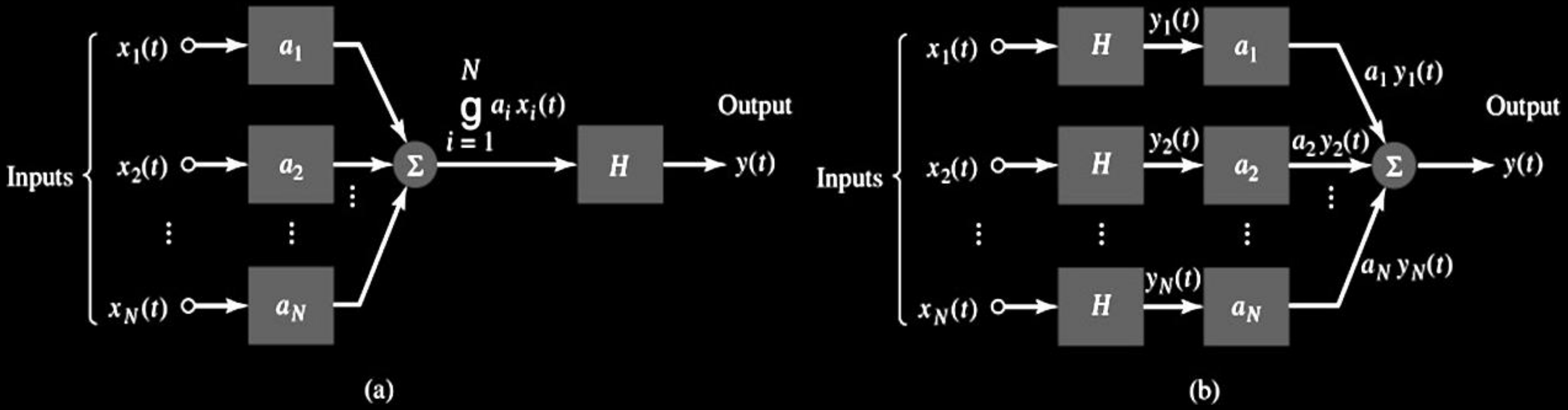
$$y[n] = n \sum_{i=1}^{N} a_i x_i[n] = \sum_{i=1}^{N} a_i n x_i[n] = \sum_{i=1}^{N} a_i y_i[n]$$

Where, $y_i[n] = nx_i[n]$ is the output due to each input acting independently.

We thus see that the given system satisfies both superposition and homogeneity and is therefore linear.

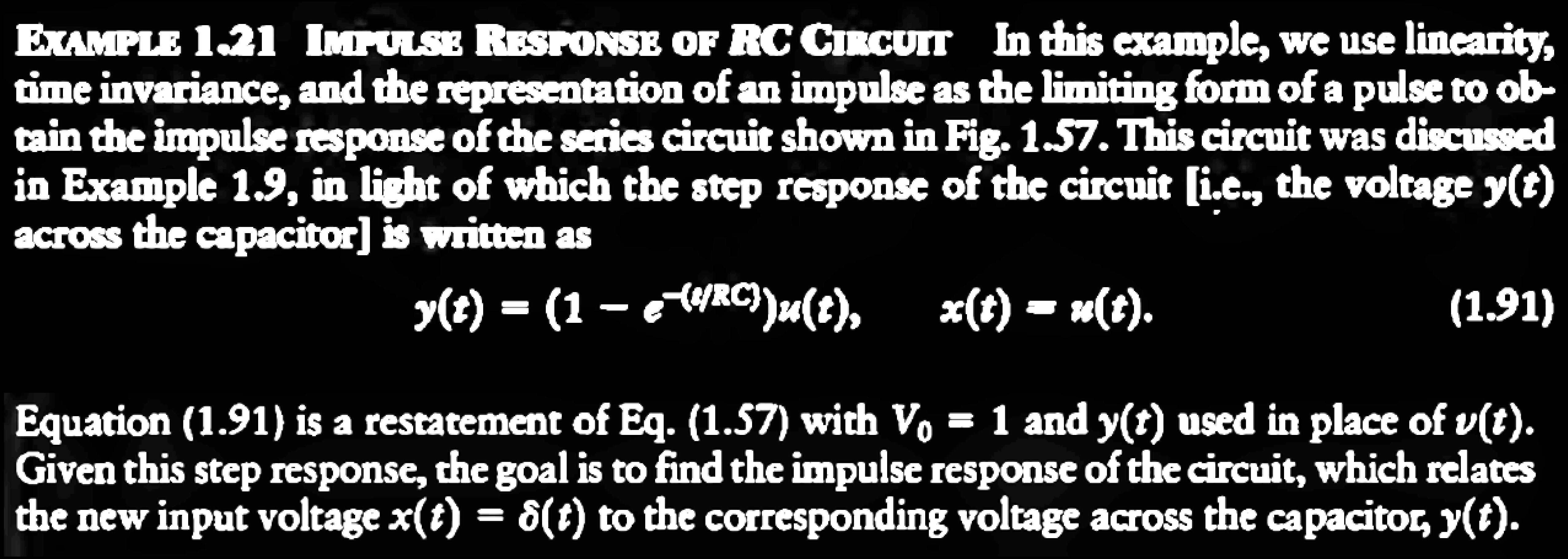
Figure 1.56 (p. 64)

The linearity property of a system. If these two configurations produce the same output $y(t)$, the operator H is linear.



(a) The combined operation of amplitude-scaling and summation precedes the operator H for multiple inputs.

(b) The operator H precedes amplitude scaling for each input; the resulting outputs are summed to produce the overall



EXAMPLE 1.21 IMPULSE RESPONSE OF *RC* CIRCUIT In this example, we use linearity, time invariance, and the representation of an impulse as the limiting form of a pulse to obtain the impulse response of the series circuit shown in Fig. 1.57. This circuit was discussed in Example 1.9, in light of which the step response of the circuit [i.e., the voltage $y(t)$ across the capacitor] is written as

$$y(t) = (1 - e^{-(t/RC)})u(t), \qquad x(t) = u(t). \tag{1.91}$$

Equation (1.91) is a restatement of Eq. (1.57) with $V_0 = 1$ and $y(t)$ used in place of $v(t)$. Given this step response, the goal is to find the impulse response of the circuit, which relates the new input voltage $x(t) = \delta(t)$ to the corresponding voltage across the capacitor, $y(t)$.

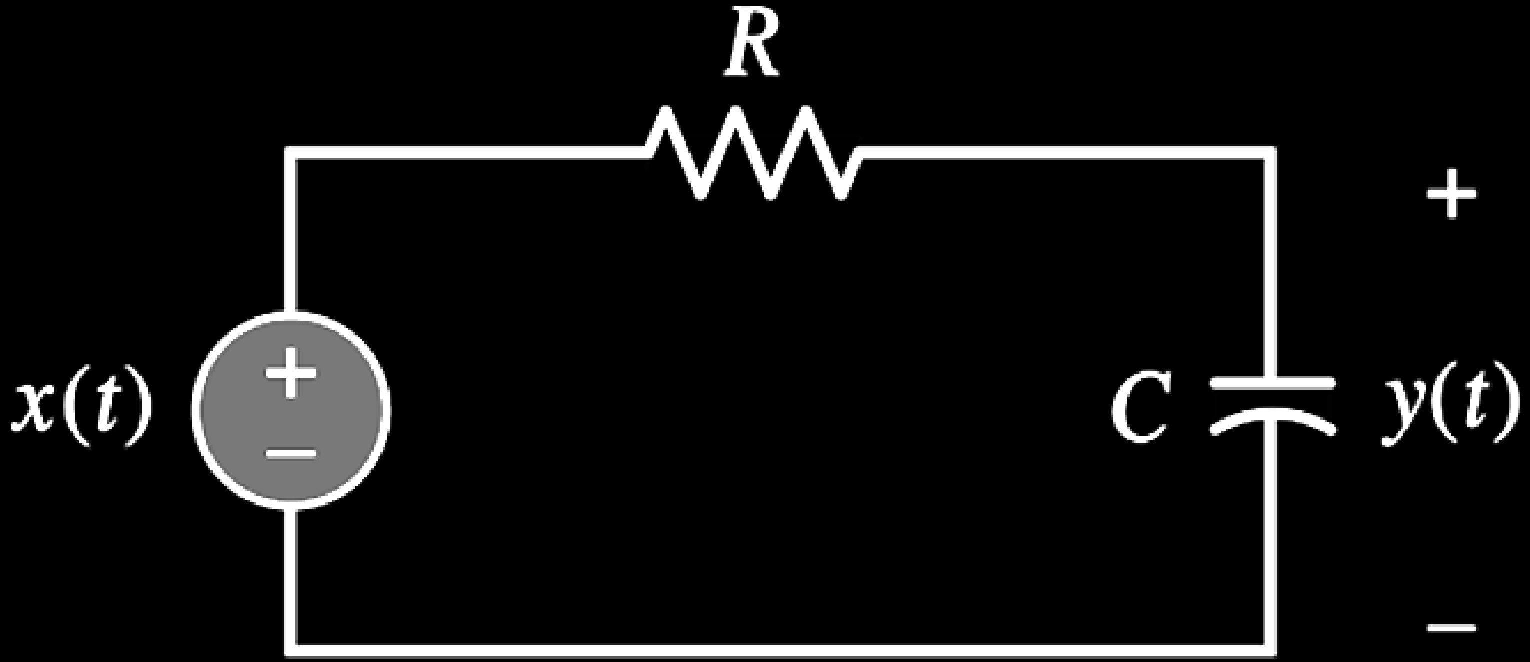
Example 1.21 Impulse Response of RC Circuit

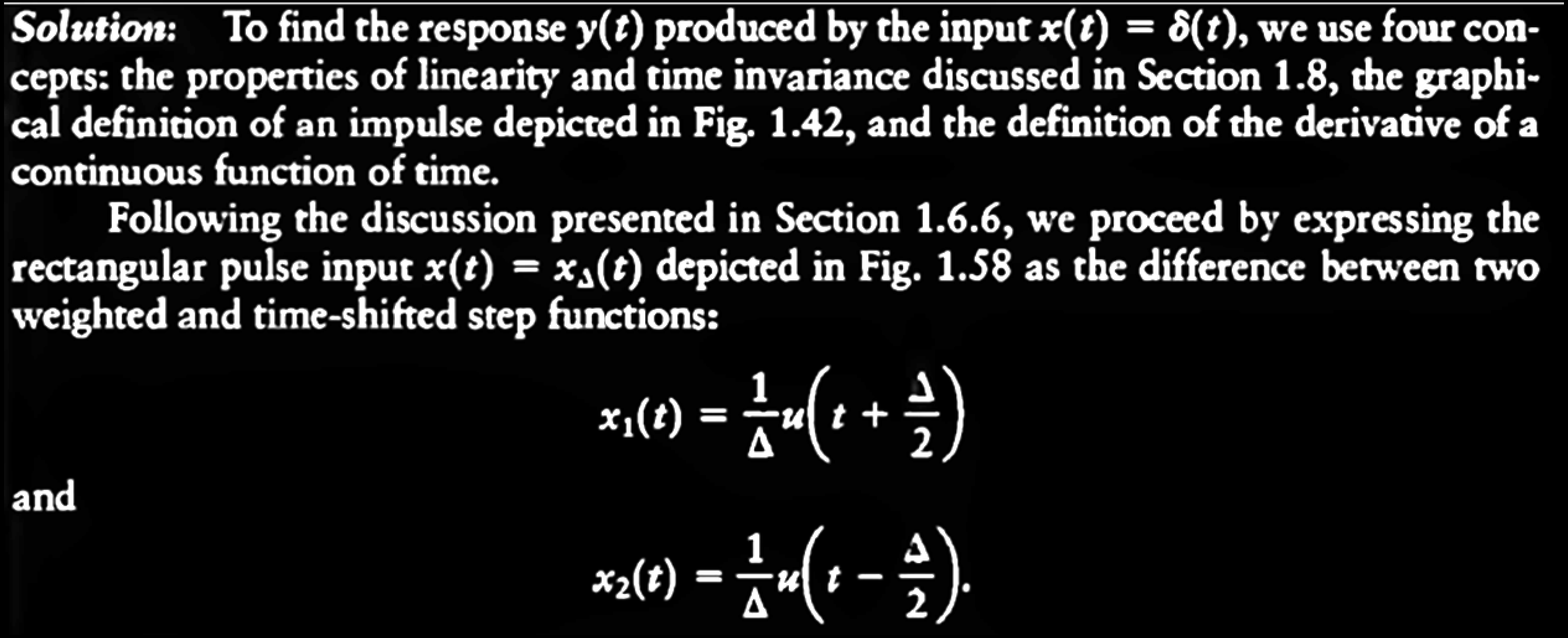
In this example, we use linearity, time invariance, and the representation of an impulse as the limiting form of a pulse to obtain the impulse response of the series circuit shown in Fig. 1 .57. This circuit was discussed in Example 1 .9, in light of which the step response of the circuit [i.e., the voltage y(t) across the capacitor] is written as

$$y(t) = \left(1 - e^{-\left(\frac{t}{RC}\right)}\right)u(t) \qquad x(t) = u(t) \qquad (1.91)$$

Equation (1.91) is a restatement of Eq. (1.57) with $V_0 = 1$ and $y(t)$ used in place of $v(t)$. Given this step response, the goal is to find the impulse response of the circuit, which relates the new input voltage $x(t) = \delta(t)$ to the corresponding voltage across the capacitor, $y(t)$.

Figure 1.57 (p. 66)
RC circuit for Example 1.20, in which we are given the capacitor voltage $y(t)$ in response to the step input $x(t) = y(t)$ and the requirement is to find $y(t)$ in response to the unit-impulse input $x(t) = \delta(t)$.





Solution: To find the response $y(t)$ produced by the input $x(t) = \delta(t)$, we use four concepts: the properties of linearity and time invariance discussed in Section 1.8, the graphical definition of an impulse depicted in Fig. 1.42, and the definition of the derivative of a continuous function of time.

Following the discussion presented in Section 1.6.6, we proceed by expressing the rectangular pulse input $x(t) = x_\Delta(t)$ depicted in Fig. 1.58 as the difference between two weighted and time-shifted step functions:

$$x_1(t) = \frac{1}{\Delta}u\left(t + \frac{\Delta}{2}\right)$$

and

$$x_2(t) = \frac{1}{\Delta}u\left(t - \frac{\Delta}{2}\right).$$

Solution 1.21.

To find the response y(t) produced by the input x(t) = 5(f), we use four concepts: the properties of linearity and time invariance discussed in Section 1.8, the graphical definition of an impulse depicted in Fig. 1.42, and the definition of the derivative of a continuous function of time.

Following the discussion presented in Section 1.6.6, we proceed by expressing the rectangular pulse input $x(t) = x_{\Delta}(t)$ depicted in Fig. 1.58 as the difference between two weighted and time-shifted step functions:

$$x_1(t) = \frac{1}{\Delta} u\left(t + \frac{\Delta}{2}\right)$$

And

$$x_2(t) = \frac{1}{\Delta} u\left(t - \frac{\Delta}{2}\right)$$

Let $y_1(t)$ and $y_2(t)$ be the responses of the RC circuit to the step functions $x_1(t)$ and $x_2(t)$, respectively. Then, applying the time-invariance property to Eq. (1.91), we have

$$y_1(t) = \frac{1}{\Delta}\left(1 - e^{-(t+\Delta/2)/(RC)}\right)u\left(t + \frac{\Delta}{2}\right), \qquad x(t) = x_1(t),$$

And

$$y_2(t) = \frac{1}{\Delta}\left(1 - e^{-(t-\Delta/2)/(RC)}\right)u\left(t - \frac{\Delta}{2}\right), \qquad x(t) = x_2(t).$$

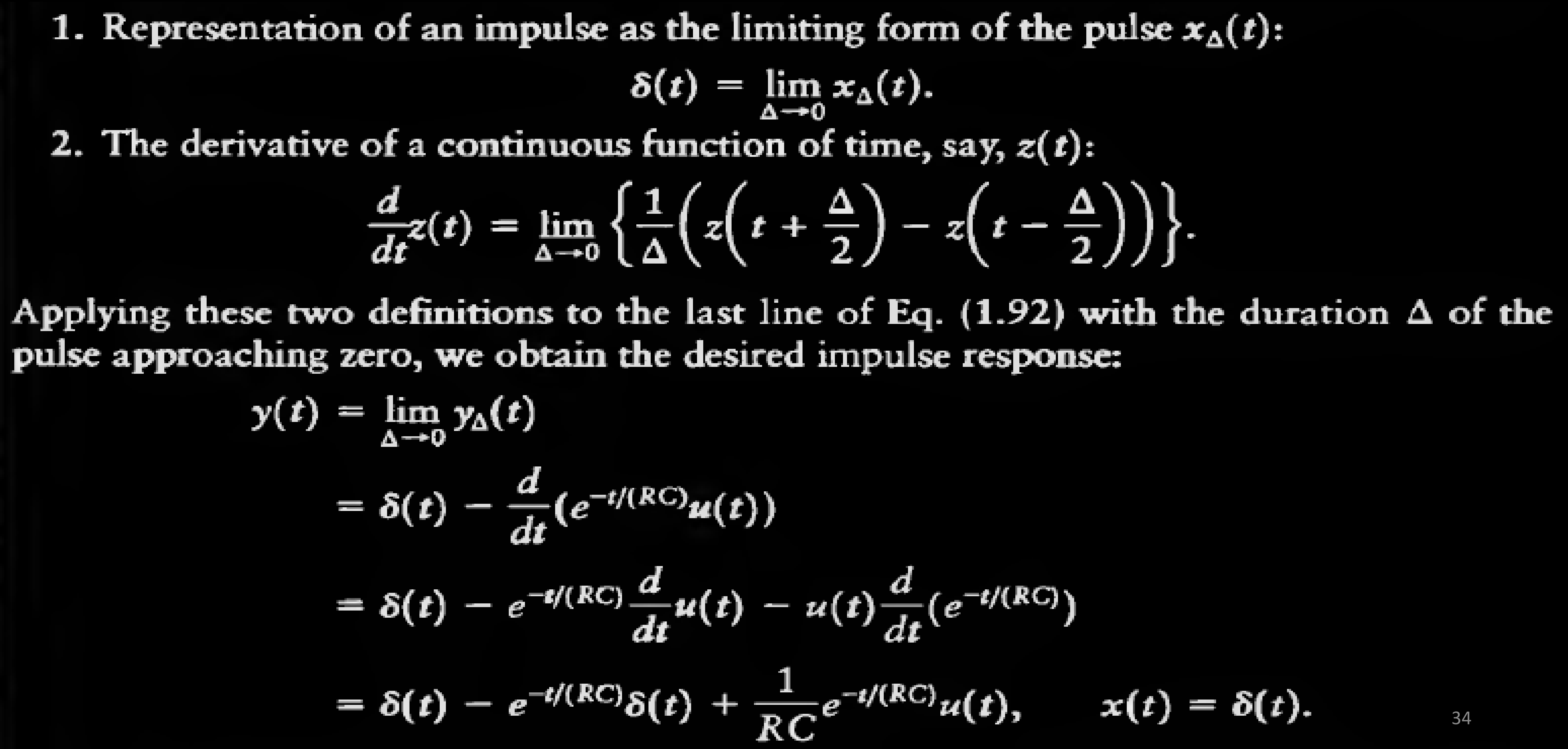
Next, recongised that

$$x_\Delta(t) = x_1(t) - x_2(t)$$

we invoke the property of linearity to express the corresponding response of the RC circuit as

$$\begin{aligned} y_\Delta(t) &= \frac{1}{\Delta}\left(1 - e^{-(t+\Delta/2)/(RC)}\right)u\left(t + \frac{\Delta}{2}\right) - \frac{1}{\Delta}\left(1 - e^{-(t-\Delta/2)/(RC)}\right)u\left(t - \frac{\Delta}{2}\right) \\ &= \frac{1}{\Delta}\left(u\left(t + \frac{\Delta}{2}\right) - u\left(t - \frac{\Delta}{2}\right)\right) \qquad (1.92) \\ &\quad - \frac{1}{\Delta}\left(e^{-(t+\Delta/2)/(RC)}u\left(t + \frac{\Delta}{2}\right) - e^{-(t-\Delta/2)/(RC)}u\left(t - \frac{\Delta}{2}\right)\right). \end{aligned}$$

All that remains for us to do is to determine the limiting form of Eq. (1.92) as the duration Δ of the pulse approaches zero. Toward that end, we invoke the following two definitions:



1. Representation of an impulse as the limiting form of the pulse $x_\Delta(t)$:

$$\delta(t) = \lim_{\Delta \to 0} x_\Delta(t).$$

2. The derivative of a continuous function of time, say, $z(t)$:

$$\frac{d}{dt}z(t) = \lim_{\Delta \to 0}\left\{\frac{1}{\Delta}\left(z\left(t + \frac{\Delta}{2}\right) - z\left(t - \frac{\Delta}{2}\right)\right)\right\}.$$

Applying these two definitions to the last line of Eq. (1.92) with the duration Δ of the pulse approaching zero, we obtain the desired impulse response:

$$\begin{aligned} y(t) &= \lim_{\Delta \to 0} y_\Delta(t) \\ &= \delta(t) - \frac{d}{dt}\left(e^{-t/(RC)}u(t)\right) \\ &= \delta(t) - e^{-t/(RC)}\frac{d}{dt}u(t) - u(t)\frac{d}{dt}\left(e^{-t/(RC)}\right) \\ &= \delta(t) - e^{-t/(RC)}\delta(t) + \frac{1}{RC}e^{-t/(RC)}u(t), \qquad x(t) = \delta(t). \end{aligned}$$

Note that in the second line we applied the rule for differentiating the product of two time functions $u(t)$ and $e^{-t/RC}$. Finally, since $\delta(t)$ is confined to the origin and $e^{-t/RC} = 1$ at t = 0, the terms $\delta(t)$ and e $e^{-\frac{t}{RC}}\delta(t)$ cancel each other, and the expression for the impulse response of the RC circuit simplifies to

$$y(t) = \frac{1}{RC}e^{-t/(RC)}u(t), \qquad x(t) = \delta(t).$$

This is the required result.

Figure 1.58 (p. 66)
Rectangular pulse of unit area, which, in the limit, approaches a unit impulse as Δ→0.

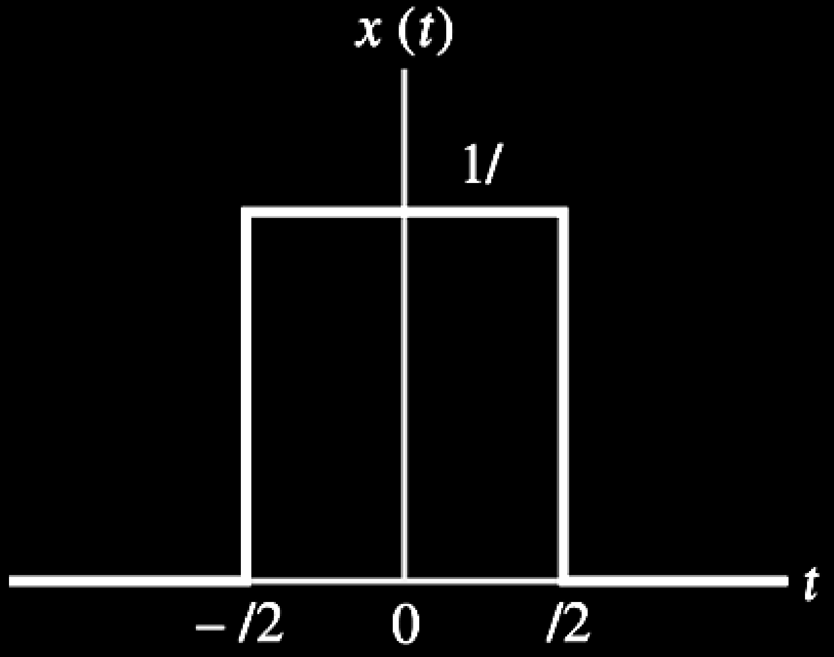
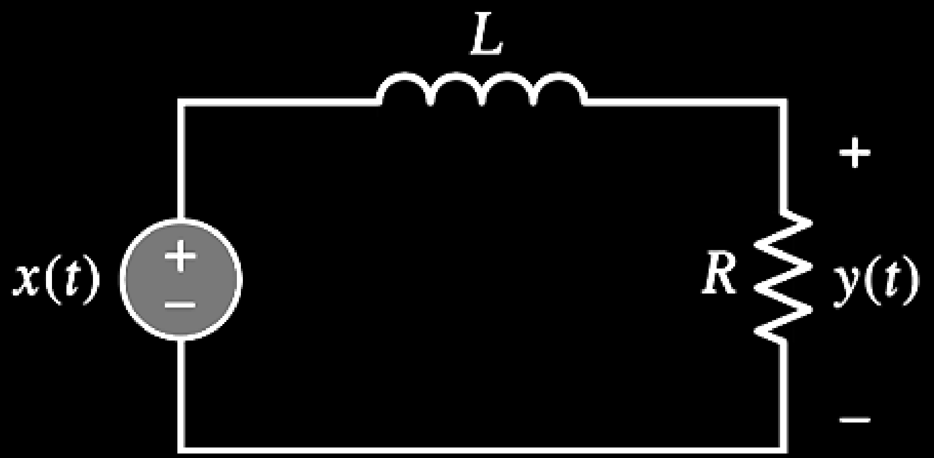


Figure 1.59 (p. 67)
LR circuit for Problem 1.37.



1.9 Noise

- The term noise is used to designate unwanted signals that tend to disturb the operation of a system and over which we have incomplete control
- For example, in a communication system, we have the following two broadly defined categories of noise:
  - **External sources of noise:** e.g. atmospheric noise, galactic noise and human-made noise falling inside the operating frequency range of the system
  - **Internal sources of noise:** noise that arises from spontaneous fluctuations of current or voltage signal in electrical circuits, referred to as electrical noise
- Figure 1.60 shows a sample waveform of electrical noise generated by a thermionic diode noise generator, which consists of a vacuum-tube diode with a heated cathode and a plate (the anode) that collects electrons emitted by cathode

1.9.1 Thermal Noise

- The thermal noise voltage $v(t)$ appearing across the terminals of a resistor has the following two characteristics:
- A time average value over the total time interval 2T defined by

$$\bar{v} = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} v(t)\, dt, \qquad (1.94)$$

- A time-average-squared value, defined by

$$\overline{v^2} = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} v^2(t)\, dt. \qquad (1.95)$$

In the limit, as T approaches infinity, we have

$$\overline{v^2} = 4kT_{abs}R\Delta f \qquad \text{volts}^2 \qquad (1.96)$$

- where k is Boltzmann's constant, equal to $1.38\text{x}10^{-23}$ joule per degree kelvin, T_{abs} is the absolute temperature in degrees kelvin, R is the resistance in ohms,

- There are therefore two operating factors that affect the available noise power $\overline{v^2} = 4kT_{abs}R\Delta f$
    1. The temperature at which the resistor R is maintained
    2. The width of the frequency band over which the noise voltage across resistor is measured

Click for Thermal Noise Animation

1.9.2 Other Sources of Electrical Noise

- Another common source of electrical noise is shot noise, which arises in electronic devices such as diodes and transistors because of the discrete nature of current flow in those devices.
- The total current flowing through the photodetector may be modeled as an infinite sum of current pulses as shown by

$$x(t) = \sum_{k=-\infty}^{\infty} h(t - \tau_k) \qquad (1.98)$$

- Where $h(t - \tau_k)$ is the current pulse generated at time τ_k

Figure 1.60 (p. 68) Sample waveform of electrical noise generated by a thermionic diode with a heated cathode.

Note that the time-averaged value of the noise voltage displayed is approximately zero.