Chapter 8: Finite Volume Method for Unsteady Flows

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8.1 Introduction

- The conservation law for the transport of a scalar in an unsteady flow has the general form

$$\frac{\partial}{\partial t}(\rho\phi) + div(\rho\mathbf{u}\phi) = div(\Gamma\, grad\, \phi) + S_\phi \tag{8.1}$$

by replacing the volume integrals of the convective and diffusive terms with surface integrals as before (see section 2.5) and changing the order of integration in the rate of change term we obtain:

$$\begin{aligned}\int_{CV}\left(\int_t^{t+\Delta t}\frac{\partial}{\partial t}(\rho\phi)dt\right)dV + \int_t^{t+\Delta t}\left(\int_A n\cdot(\rho\mathbf{u}\phi)dA\right)dt \\ = \int_t^{t+\Delta t}\left(\int_A n\cdot(\Gamma\, grad\, \phi)dA\right)dt + \int_t^{t+\Delta t}\int_{CV} S_\phi dVdt\end{aligned} \tag{8.2}$$

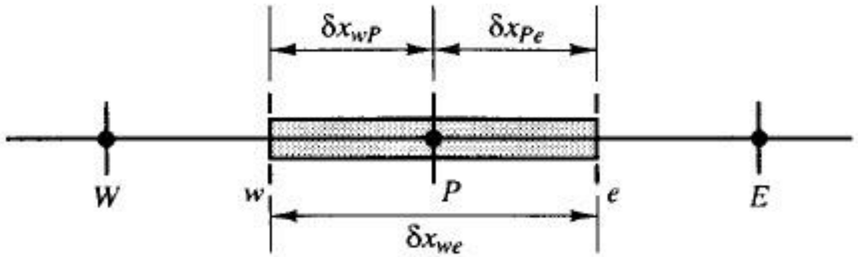
Introduction

- Unsteady one-dimensional heat conduction is governed by the equation

$$\rho c \frac{\partial T}{\partial t} = \frac{\partial}{\partial x}\left(k \frac{\partial T}{\partial x} \right) + S \qquad (8.3)$$

- In addition to usual variables we have *c*, the specific heat of material (J/kg/K).

Fig. 8.1



- Consider the one-dimensional control volume in Figure 8.1. Integration of equation (8.3) over the control volume and over a time interval from *t* to *t+Δt* gives

$$\int_{t}^{t+\Delta t} \int_{CV} \rho c \frac{\partial T}{\partial t} dV dt = \int_{t}^{t+\Delta t} \int_{CV} \frac{\partial}{\partial x}\left(k \frac{\partial T}{\partial x} \right) dV dt + \int_{t}^{t+\Delta t} \int_{CV} s dV dt \qquad (8.4)$$

- This may be written as

$$\int_{w}^{e} \left[\int_{t}^{t+\Delta t} \rho c \frac{\partial T}{\partial t} dt \right] dV = \int_{t}^{t+\Delta t} \left[\left(kA \frac{\partial T}{\partial x} \right)_e - \left(kA \frac{\partial T}{\partial x} \right)_w \right] dt + \int_{t}^{t+\Delta t} \bar{S} \Delta V dt \qquad (8.5)$$

- The left hand side can be written as

$$\int_{CV}\left[\int_{t}^{t+\Delta t}\rho c\frac{\partial T}{\partial t}dt\right]dV=\rho c\left(T_P-T_P^0\right)\Delta V \qquad (8.6)$$

- In equation(8.6) superscript ‘o’ refers to temperatures at time *t*.

Temperatures at time level $t+\Delta t$ are not superscripted.

Eqn(8.6) could also be obtained by substituting

$$\frac{\partial T}{\partial t}=\frac{T_P-T_P^0}{\Delta t}$$

So, first order (backward) differencing scheme has been used. If we apply central differencing to rhs of eqn (8.5),

$$\rho c\left(T_P-T_P^0\right)\Delta V=\int_{t}^{t+\Delta t}\left[\left(k_eA\frac{T_E-T_P}{\partial x_{PE}}\right)-\left(k_wA\frac{T_P-T_W}{\partial x_{WP}}\right)\right]dt+\int_{t}^{t+\Delta t}\bar{S}\Delta V dt \qquad (8.7)$$

- To calculate the integrals we have to make an assumption about the variation of T_P, T_E and T_W with time, we could use temperatures
  - at time t, or
  - at time $t+\Delta t$
  - or combination of both.

Integral of temperature T_P with respect to time can be written as;

$$I_T = \int_t^{t+\Delta t} T_P \, dt = \left[\theta T_P + (1-\theta) T_P^0 \right] \Delta t \qquad (8.8)$$

Hence

θ	0	$1/2$	1
I_T	$T_P^0 \Delta t$	$\frac{1}{2}\left(T_P + T_P^0\right)\Delta t$	$T_P \Delta t$

θ = a weighting parameter between zero and one.

- Using formula (8.8) for T_W and T_E in equation (8.7), and dividing by $A\Delta t$ throughout, we have

$$\rho c\left(\frac{T_P - T_P^0}{\Delta t}\right)\Delta x = \theta\left[\frac{k_e\left(T_E - T_P\right)}{\delta x_{PE}} - \frac{k_w\left(T_P - T_W\right)}{\delta x_{WP}}\right] + \left(1-\theta\right)\left[\frac{k_e\left(T_E^0 - T_P^0\right)}{\delta x_{PE}} - \frac{k_w\left(T_P^0 - T_W^0\right)}{\delta x_{WP}}\right] + \bar{S}\Delta x \qquad (8.9)$$

which may be re-arranged to give

$$\left[\rho c\frac{\Delta x}{\Delta t} + \theta\left(\frac{k_e}{\delta x_{PE}} + \frac{k_w}{\delta x_{WP}}\right)\right]T_P = \frac{k_e}{\delta x_{PE}}\left[\theta T_E + \left(1-\theta\right)T_E^0\right] + \frac{k_w}{\delta x_{WP}}\left[\theta T_W + \left(1-\theta\right)T_W^0\right] + \left[\rho c\frac{\Delta x}{\Delta t} - \left(1-\theta\right)\frac{k_e}{\delta x_{PE}} - \left(1-\theta\right)\frac{k_w}{\delta x_{WP}}\right]T_P^0 + \bar{S}\Delta x \qquad (8.10)$$

- The source term is linearized as $b=S_u+S_P T_P$. Now we identify the coefficients of T_W and T_E as a_W and a_E and write equation (8.10) in familiar standard form:

$$a_p T_P = \theta a_W T_W + \theta a_E T_E + (1-\theta) a_W T_W^0 + (1-\theta) a_E T_E^0 + \left[a_P^0 - (1-\theta) a_W - (1-\theta) a_E \right] T_P^0 + b \quad (8.11)$$

where

and

$$a_P = \theta\left(a_W + a_E\right) + a_P^0 - \theta S_P$$

with

$$a_P^0 = \rho c \frac{\Delta x}{\Delta t}$$

a_W	a_E	b
$\dfrac{k_w}{\delta x_{WP}}$	$\dfrac{k_e}{\delta x_{PE}}$	$S_u + (1-\theta) S_P T_P^0$

For $\theta = 0$ → explicit scheme

$0 < \theta < 1$ → implicit scheme, for $\theta = 0.5$ → Crank-Nicolson scheme

$\theta = 1$ → Fully implicit scheme

8.2.1 Explicit scheme

- In the explicit scheme the source term is linearized as $b=S_u+S_pT_p{}^0$. Now the substitution of $\theta = 0$ into (8.11) gives the **explicit discretisation** of the unsteady conductive heat transfer equation:

$$a_P T_P = a_W T_W^0 + a_E T_E^0 + \left[a_P^0 - \left(a_W + a_E - S_P \right) \right] T_P^0 + S_u \qquad (8.12)$$

where

$$a_P = a_P^0$$

and

$$a_P^0 = \rho c \frac{\Delta x}{\Delta t}$$

a_W	a_E
$\dfrac{k_w}{\delta x_{WP}}$	$\dfrac{k_e}{\delta x_{PE}}$

The right hand side of eqn (8.12) only contains values at the old time step so the left hand side can be calculated by forward marching in time. The scheme is based on backward differencing, and is of first order accurate.

- All coefficients should be positive → $a_P{}^0 - a_W - a_E > 0$

or if $k = const.$ and $\delta x_{PE} = \delta x_{WP} = \Delta x$, this condition can be written as

$$\rho c \frac{\Delta x}{\Delta t} > \frac{2k}{\Delta x} \qquad (8.13a)$$

Or

$$\Delta t < \rho c \frac{(\Delta x)^2}{2k} \qquad (8.13b)$$

This inequality sets a stringent maximum limit to the time step size and represents a serious limitation for the explicit scheme.

Not recommended for the explicit scheme problems.

8.2.2 Crank – Nicolson scheme

- The Crank – Nicolson method results from setting θ = ½ in eqn. (8.11). Now the discretised unsteady heat conduction equation is

$$a_p T_P = \frac{1}{2} a_W T_W + \frac{1}{2} a_E T_E + \frac{1}{2} a_W T_W^0 + \frac{1}{2} a_E T_E^0 + \left[a_P^0 - \frac{a_W}{2} - \frac{a_E}{2} \right] T_P^0 + b \qquad (8.14)$$

where $$a_P = \frac{1}{2}\left(a_W + a_E\right) + a_P^0 - \frac{1}{2} S_P$$

and $$a_P^0 = \rho c \frac{\Delta x}{\Delta t}$$

a_W	a_E	b
$\dfrac{k_w}{\delta x_{WP}}$	$\dfrac{k_e}{\delta x_{PE}}$	$S_u + \dfrac{1}{2} S_P T_P^0$

- Schemes with $½ \le \theta \le 1$ are unconditionally stable for all values of time step Δt.
- However, for physically realistic results all coefficients should be positive. Then, coefficients of T_P^0 should be positive, or

$$a_P^0 > \left(\frac{a_W + a_E}{2} \right)$$

which leads to

$$\Delta t < \rho c \frac{\Delta x^2}{k} \quad (8.15)$$

This time step limitation is only slightly less restrictive than (8.13) associated with the explicit method

The method is based on central differencing → Second order accurate in time

Is more accurate than explicit method

The fully implicit scheme

- When θ =1 we obtain the fully implicit scheme. The source term is linearized as $b=S_u+S_P T_P$. The discretised equation is

$$a_p T_P = a_W T_W + a_E T_E + a_P^0 T_P^0 + S_u \qquad (8.16)$$

where

$$a_P = a_P^0 + a_W + a_E - S_P$$

and

$$a_P^0 = \rho c \frac{\Delta x}{\Delta t}$$

with

a_W	a_E
$\dfrac{k_w}{\delta x_{WP}}$	$\dfrac{k_e}{\delta x_{PE}}$

Both sides of the equation contains temperatures at the new time step, and a system of algebraic equations must be solved at each time level

- is unconditionally stable for any Δt
- is only first order accurate in time

→ small time steps are needed to ensure accuracy of results

- **Example 8.1**

A thin plate is initially at a uniform temperature of 200 ºC. At a certain time *t*=0 the temperature of the east side of the plate is suddenly reduced to 0ºC. The other surface is insulated. Use the explicit finite volume method in conjunction with a suitable time step size to calculate the transient temperature distribution of the slab and compare it with the analytical solution at time (i) t = 40s, (ii) t = 80s and (iii) t = 120s. Recalculate the numerical solution using a time step size equal to the limit given by (8.13) for t = 40s and compare the results with the analytical solution. The data are: plate thickness L=2cm, thermal conductivity k = 10 W/m/K and $\rho c = 10 \times 10^6$ J/m^3/K.

Solution: the one–dimensional transient heat conduction eqn is

$$\rho c \frac{\partial T}{\partial t} = \frac{\partial}{\partial x}\left(k \frac{\partial T}{\partial x}\right) + S$$

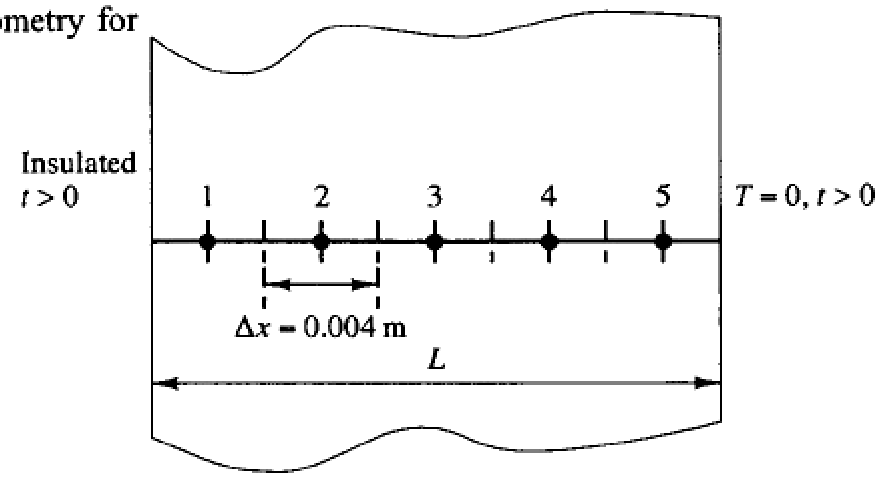
The initial and boundary conditions:

$$T = 200 \quad \text{at} \quad t = 0$$

$$\frac{\partial T}{\partial t} = 0 \quad \text{at} \quad x = 0, t > 0$$

$$T = 0 \quad \text{at} \quad x = L, t > 0$$

Fig. 8.2 Geometry for Example 8.1



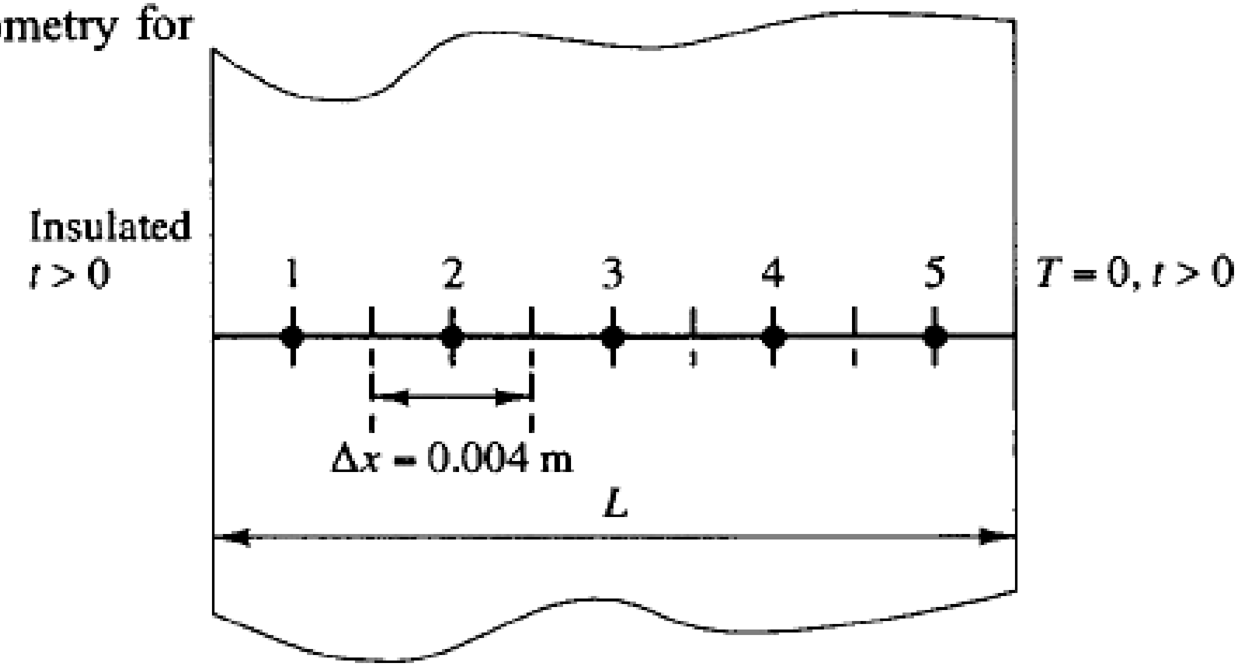
- The analytical solution is given in Ozisik (1985) as

$$\frac{T(x,t)}{200} = \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{2n-1}\exp\left(-\alpha\lambda_n^2 t\right)\cos\left(\lambda_n x\right) \qquad (8.18)$$

where $\lambda_n = \dfrac{(2n-1)\pi}{2L}$ and $\alpha = k/\rho c$

The numerical solution with the explicit method is generated by dividing the domain width L into five equal control volumes with Δx = 0.004m. The resulting one-dimensional grid is shown in Figure 8.2.

Fig. 8.2 Geometry for Example 8.1



- The time step for the explicit method is subject to the condition that

$$\Delta t < \frac{\rho c\left(\Delta x\right)^2}{2k}$$

$$\Delta t < \frac{10\times10^6\left(0.004\right)^2}{2\times10}$$

$$\Delta t < 8s$$

Fig. 8.3 Comparison of numerical and analytical solutions at different times

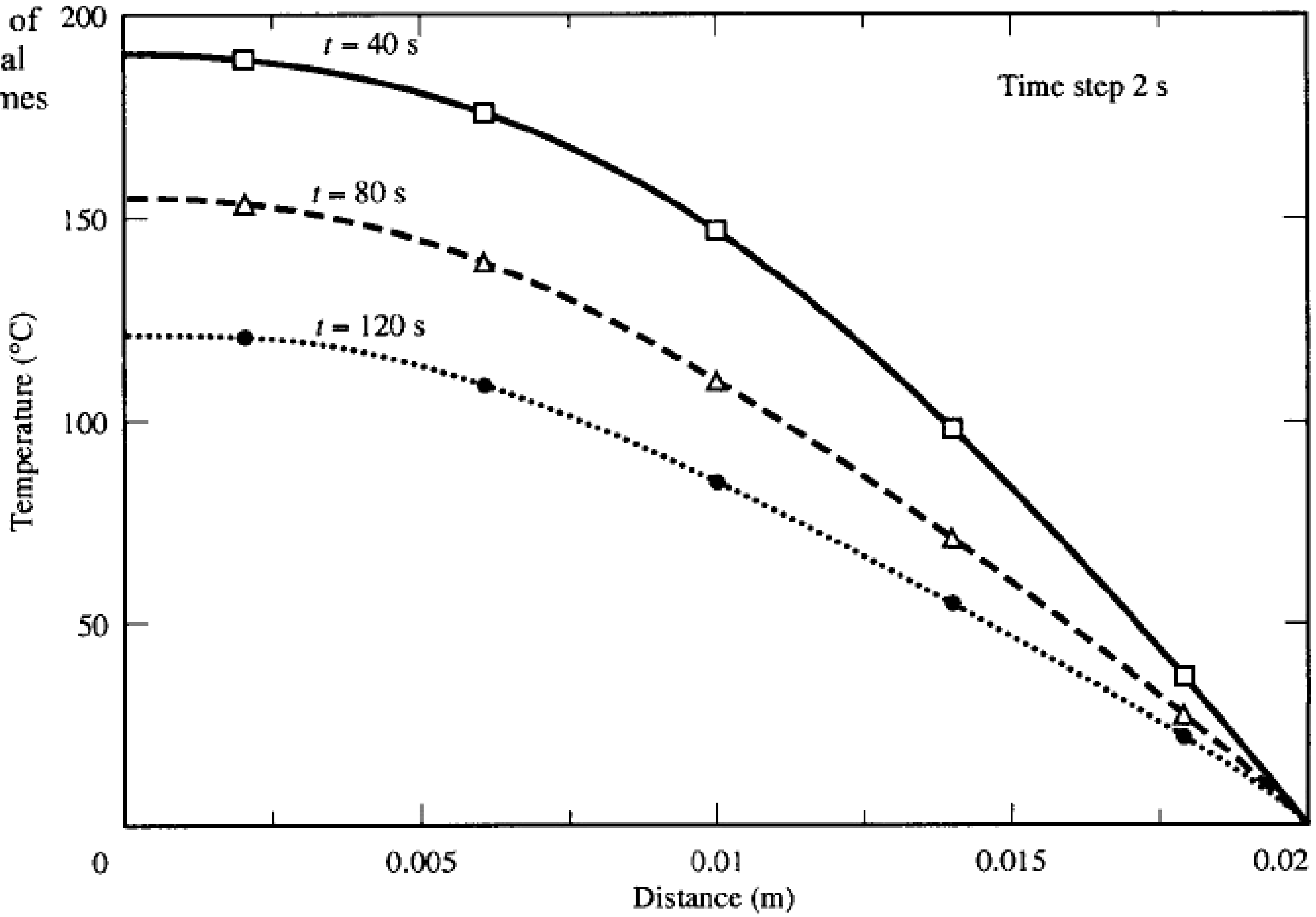
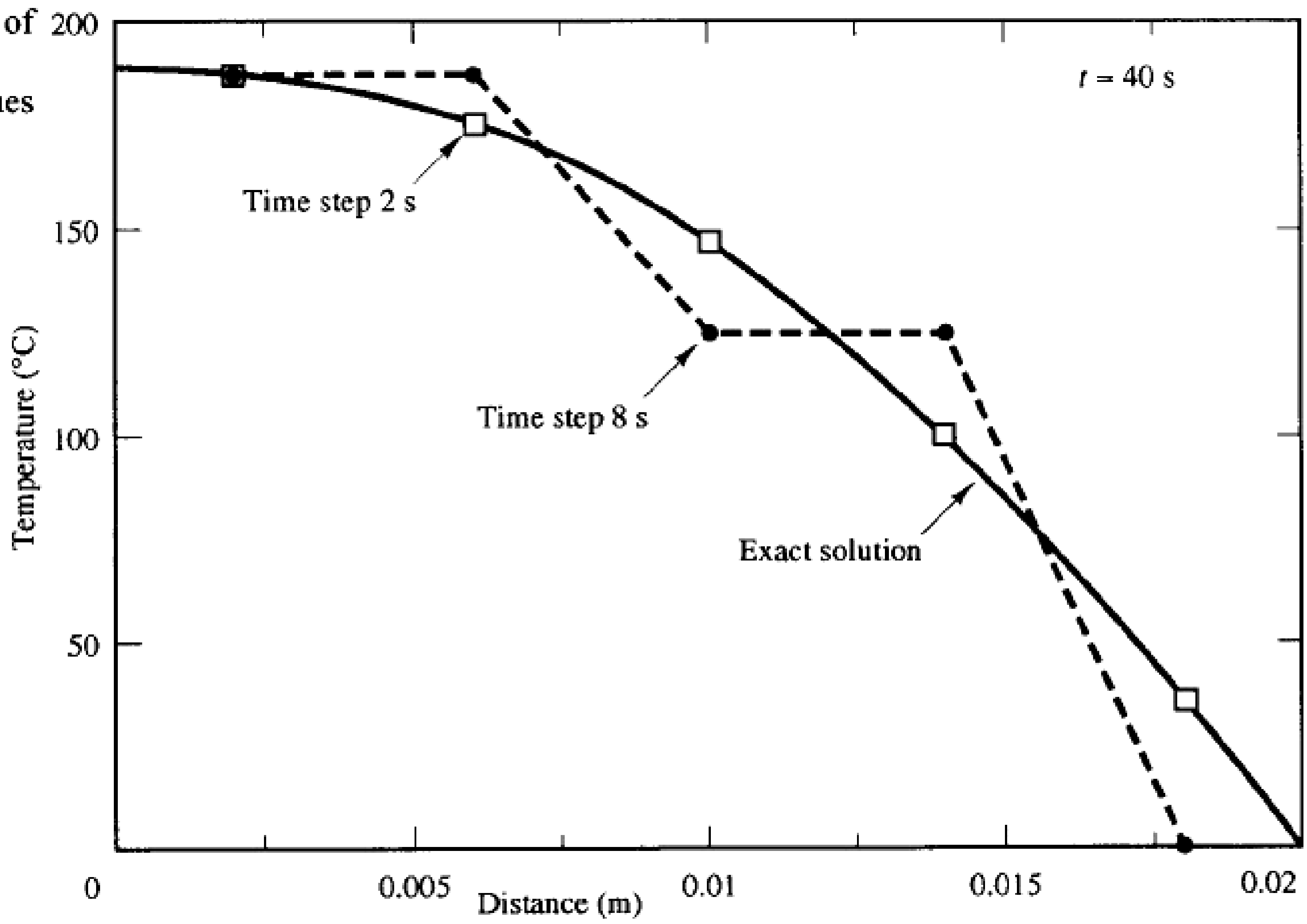


Fig. 8.4 Comparison of results obtained using different time step values



- **Example 8.2** solve the problem of Example 8.1 again, using the fully implicit method and compare the explicit and implicit method solutions for a time step of 8 s.

Table 8.4

Point	*Time* = 40 *s*			*Time* = 80 *s*			*Time* = 120 *s*		
	Numerical	Analytical	% error	Numerical	Analytical	% error	Numerical	Analytical	% error
1	187.38	188.38	0.51	153.72	152.65	−0.70	121.52	119.87	−1.42
2	176.28	175.76	−0.29	139.79	138.36	−1.03	109.78	108.21	−1.24
3	150.04	147.13	−1.97	112.38	110.63	−1.57	87.33	85.96	−1.59
4	103.69	99.50	−4.20	73.09	71.56	−2.13	56.20	55.25	−1.71
5	37.51	35.38	−6.02	25.38	24.77	−2.46	19.39	19.05	−1.78

Fig. 8.5 Comparison of numerical results with the analytical solution (implicit method)

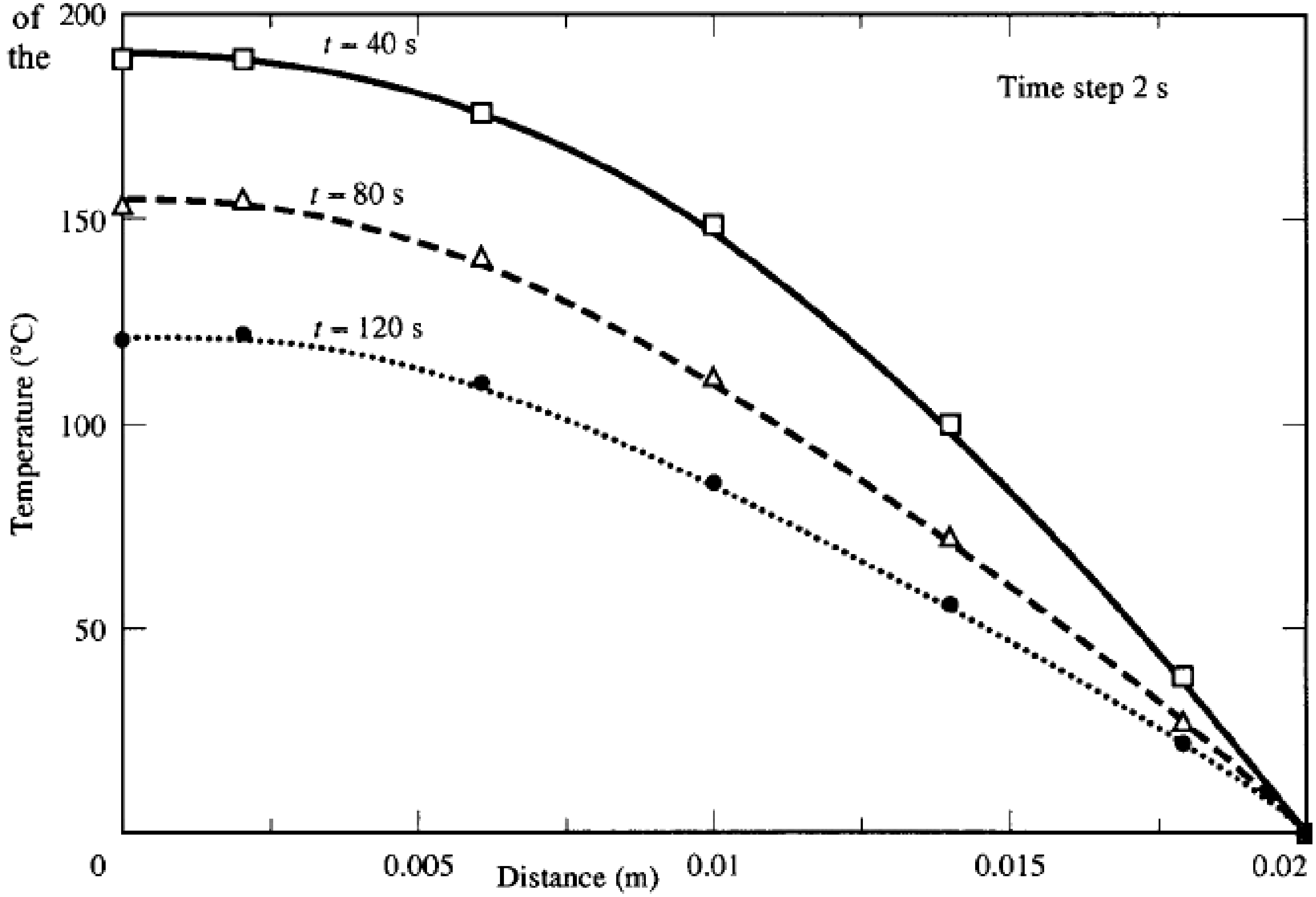
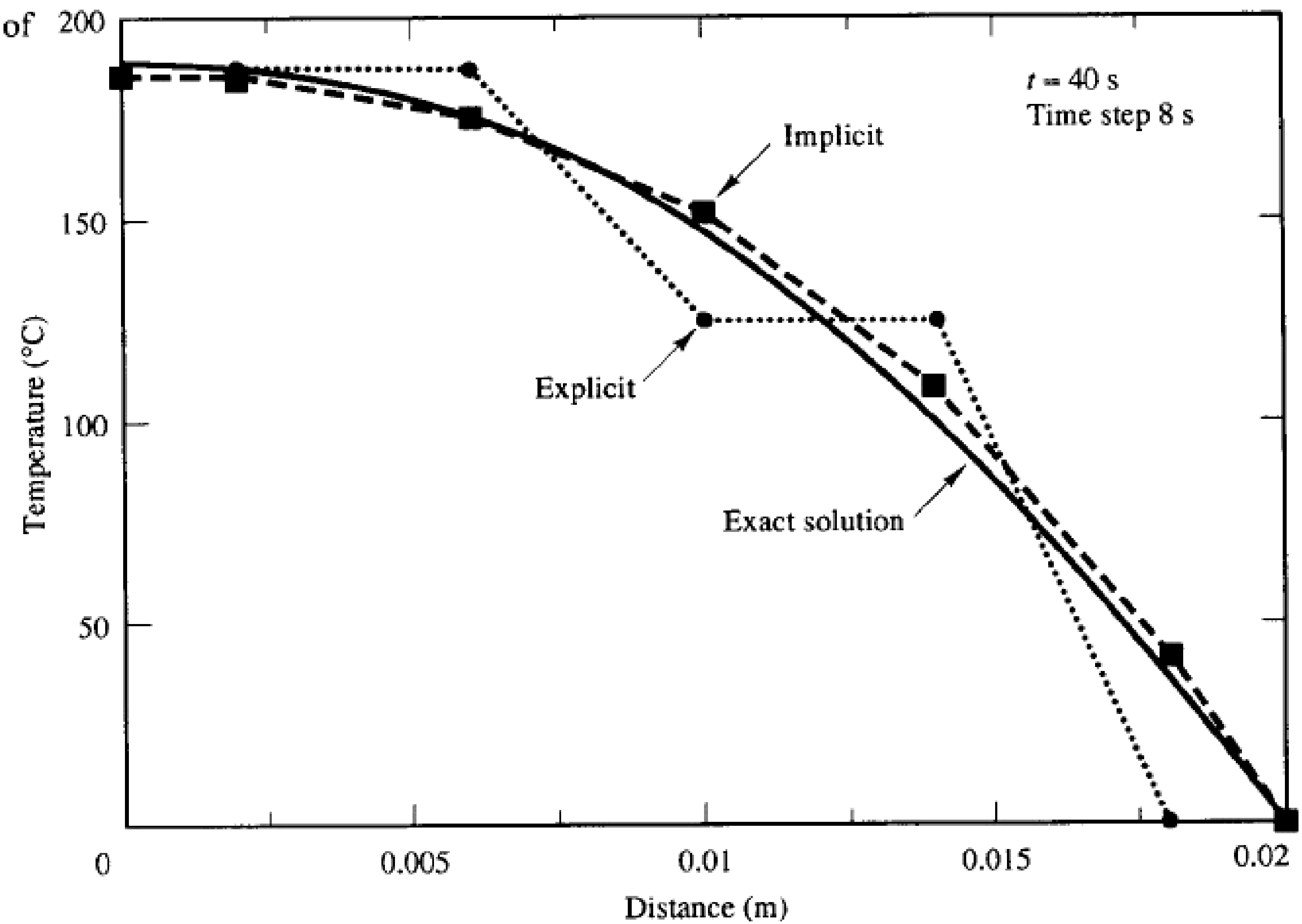


Fig. 8.6 Comparison of implicit and explicit solutions for $\Delta t = 8s$



8.4 Implicit Method for Two-and Three- dimensional Problems

- Transient diffusion equation in three-dimensions is governed by

$$\rho c \frac{\partial \phi}{\partial t} = \frac{\partial}{\partial x}\left(k\frac{\partial \phi}{\partial x}\right) + \frac{\partial}{\partial y}\left(k\frac{\partial \phi}{\partial y}\right) + \frac{\partial}{\partial z}\left(k\frac{\partial \phi}{\partial z}\right) + S \qquad (8.27)$$

A three dimensional control volume is considered for the discretisation. The resulting equation is

$$a_P\phi_P = a_W\phi_W + a_E\phi_E + a_S\phi_S + a_N\phi_N + a_B\phi_B + a_T\phi_T + a_P^0\phi_P^0 + S_u \qquad (8.28)$$

where

$$a_P = a_W + a_E + a_S + a_N + a_B + a_T + a_P^0 - S_P$$

$$a_P^0 = \rho c \frac{\Delta V}{\Delta t}$$

$S = (S_u + S_P\phi_P)$ is the linearized source

- A summary of the relevant neighbour coefficients is given below

	a_W	a_E	a_S	a_N	a_B	a_T
$1D$	$\dfrac{\Gamma_w A_w}{\delta x_{WP}}$	$\dfrac{\Gamma_e A_e}{\delta x_{PE}}$	–	–	–	–
$2D$	$\dfrac{\Gamma_w A_w}{\delta x_{WP}}$	$\dfrac{\Gamma_e A_e}{\delta x_{PE}}$	$\dfrac{\Gamma_s A_s}{\delta y_{SP}}$	$\dfrac{\Gamma_n A_n}{\delta y_{PN}}$	–	–
$3D$	$\dfrac{\Gamma_w A_w}{\delta x_{WP}}$	$\dfrac{\Gamma_e A_e}{\delta x_{PE}}$	$\dfrac{\Gamma_s A_s}{\delta y_{SP}}$	$\dfrac{\Gamma_n A_n}{\delta y_{PN}}$	$\dfrac{\Gamma_b A_b}{\delta z_{BP}}$	$\dfrac{\Gamma_t A_t}{\delta z_{PT}}$

- The following values for the volume and cell face areas apply in three cases:

	$1D$	$2D$	$3D$
ΔV	Δx	$\Delta x \Delta y$	$\Delta x \Delta y \Delta z$
$A_w = A_e$	1	Δy	$\Delta y \Delta z$
$A_n = A_s$	–	Δx	$\Delta x \Delta z$
$A_b = A_t$	–	–	$\Delta x \Delta y$

8.5 Discretisation of transient convection-diffusion equation

- The unsteady transport of a property ϕ is given by

$$\frac{\partial}{\partial t}(\rho\phi) + div(\rho\mathbf{u}\phi) = div(\Gamma\, grad\,\phi) + S_\phi \qquad (8.29)$$

Here, we quote the implicit/hybrid difference form of the transient convection-diffusion equations.

Transient three-dimensional convection-diffusion of a general property ϕ in a velocity field **u** is governed by

$$\frac{\partial(\rho\phi)}{\partial t} + \frac{\partial(\rho u\phi)}{\partial x} + \frac{\partial(\rho v\phi)}{\partial y} + \frac{\partial(\rho w\phi)}{\partial z} = \frac{\partial}{\partial x}\left(\Gamma\frac{\partial\phi}{\partial x}\right) + \frac{\partial}{\partial y}\left(\Gamma\frac{\partial\phi}{\partial y}\right) + \frac{\partial}{\partial z}\left(\Gamma\frac{\partial\phi}{\partial z}\right) + S_\phi \qquad (8.30)$$

- The discretised equation at location P with degree of implicitness θ is

$$a_p\phi_P = \theta a_W\phi_W + \theta a_E\phi_E + \theta a_S\phi_S + \theta a_N\phi_N + (1-\theta)a_W\phi_W^0 + (1-\theta)a_E\phi_E^0 + (1-\theta)a_S\phi_S^0 + (1-\theta)a_N\phi_N^0$$

$$+\left[-(1-\theta)a_W - (1-\theta)a_E - (1-\theta)a_S - (1-\theta)a_N\right]\phi_P^0 + S$$

$$S = s_C^{body}V_P + S^{trans} + S^{dc} + S^{pres} + (1-\theta)s_P^{body}\phi_P^0,$$

$S^{trans} = a_P^o\phi_P^o,\quad a_P^o = \dfrac{\rho_P^0 V_P}{\Delta t}$ (transient terms), $\quad s^{body} = s_C^{body} + s_P^{body}\phi_P$ (body forces per unit volume in the differential equation)

$$a_E = \frac{\Gamma_e\Delta y}{\Delta x_e} + \max\left[-F_e,\ 0\right],\qquad a_W = \frac{\Gamma_w\Delta y}{\Delta x_w} + \max\left[F_w,\ 0\right],\qquad a_N = \frac{\Gamma_n\Delta x}{\Delta y_n} + \max\left[-F_n,\ 0\right],\qquad a_S = \frac{\Gamma_s\Delta x}{\Delta y_s} + \max\left[F_s,\ 0\right]$$

$$a_P = \theta(a_E + a_W + a_N + a_S) + \frac{\rho_P V_P}{\Delta t} - \theta s_P^{body}\Delta x\Delta y + \Delta F,$$

$$\Delta F = F_e - F_w + F_n - F_s,$$

$$S^{pres} = \begin{cases} = -\dfrac{\partial p}{\partial x}\Delta V = -[(p_e - p_w)/\Delta x]\Delta V = -(p_e - p_w)\Delta y & \text{for x-momentum equation} \\[2ex] = -\dfrac{\partial p}{\partial y}\Delta V = -[(p_n - p_s)/\Delta y]\Delta V = -(p_n - p_s)\Delta x & \text{for y-momentum equation} \end{cases}$$

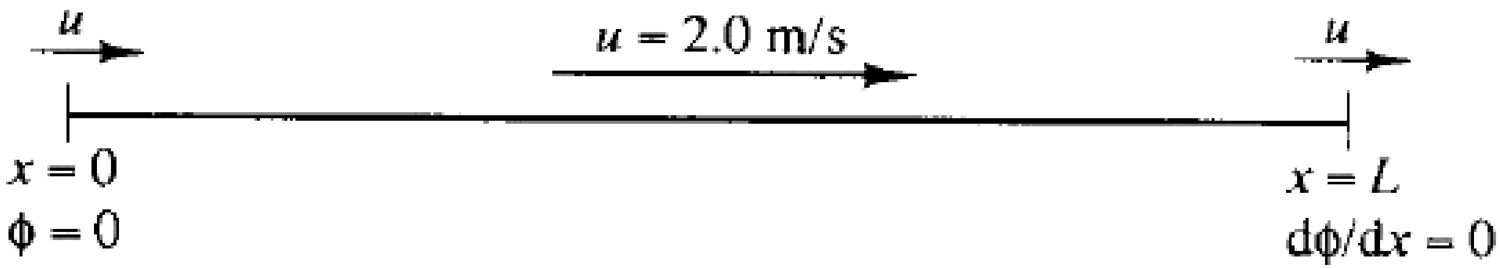
$$\begin{aligned} S^{dc} = & -\max\left[+F_e,\ 0\right]\left(\phi_e - \phi_P\right) + \max\left[-F_e,\ 0\right]\left(\phi_e - \phi_E\right) \\ & -\max\left[-F_w,\ 0\right]\left(\phi_w - \phi_P\right) + \max\left[+F_w,\ 0\right]\left(\phi_w - \phi_W\right) \\ & -\max\left[+F_n, 0\right]\left(\phi_n - \phi_P\right) + \max\left[-F_n,\ 0\right]\left(\phi_n - \phi_N\right) \\ & -\max\left[-F_s,\ 0\right]\left(\phi_s - \phi_P\right) + \max\left[+F_s,\ 0\right]\left(\phi_s - \phi_S\right) \end{aligned}$$

$F_e = \left(\rho u\right)_e\Delta y,\quad F_w = \left(\rho u\right)_w\Delta y,\quad F_n = \left(\rho v\right)_n\Delta x,\quad F_s = \left(\rho v\right)_s\Delta x,\quad$ (u_e, u_w, v_n, v_s are found by MIM method)

$\phi_e, \phi_w, \phi_n, \phi_s$ = face values found from a high order (higher than 1st order) convection scheme such as QUICK or CD

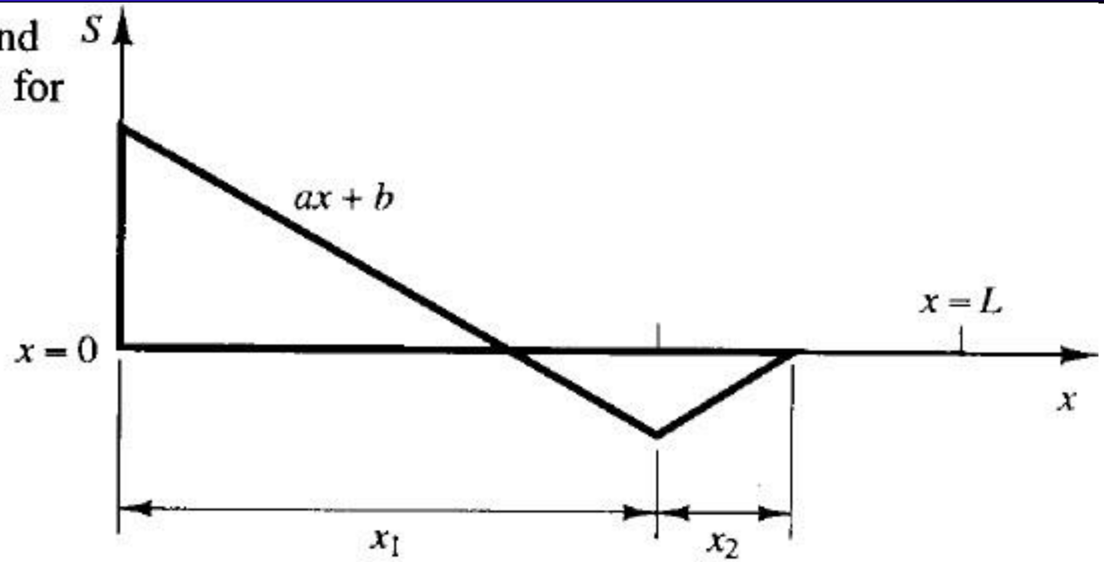
- **Example 8.3** consider convection and diffusion in the one-dimensional domain sketched in Figure 8.7. Calculate the transient temperature field if the initial temperature is zero everywhere and the boundary conditions are $\phi=0$ at $x=0$ and $\partial\phi/\partial x=0$ at $x=L$. the data are $L=1.5m$, $u=2m/s$, $\rho=1.0kg/m^3$ and $\Gamma=0.03kg/m/s$.

Fig. 8.7



the source distribution defined by Figure 8.8 applies at times t>0 with $a=-200$, $b=100$, $x_1= 0.6m$, $x_2= 0.2m$. Write a computer program to calculate the transient temperature distribution until it reaches a steady state using the implicit method for time integration and the *Hayase et al* variant of the QUICK scheme for the convective and diffusive terms and compare this result with the analytical steady state solution

Fig. 8.8 Geometry and the source distribution for Example 8.3



- **Solution** convection-diffusion of a property ϕ subjected to a distributed source term is governed by

$$\frac{\partial(\rho\phi)}{\partial t}+\frac{\partial(\rho u\phi)}{\partial x}=\frac{\partial}{\partial x}\left(\Gamma\frac{\partial\phi}{\partial x}\right)+S \qquad (8.32)$$

- We use a 45 point grid to subdivide the domain and perform all calculations with a computer program. It is convenient to use the *Hayase et al* formulation of QUICK (see section 5.9.3)since it gives a tri-diagonal system of equations which can be solved iteratively with the TDMA (see section 7.2).

The velocity is $u=2.0m/s$ and the cell width is $\Delta x=0.0333$ so $F=\rho u=2.0$ and $D=\Gamma/\delta x=0.9$ everywhere. The Hayase et al formulation gives φ at cell faces by means of the following formulae:

$$\phi_e = \phi_P + \frac{1}{8}\left(3\phi_E - 2\phi_P - \phi_W\right) \tag{8.33}$$

$$\phi_w = \phi_W + \frac{1}{8}\left(3\phi_P - 2\phi_W - \phi_{WW}\right) \tag{8.34}$$

(8.35)

- A time step $\Delta t = 0.01$ is selected, which is well within stability limit for explicit schemes

  - Start with an initial field of $\phi_P^0 = 0$ at all nodes.
  - Solve iteratively for ϕ values until a converged solution is obtained
  - Set $\phi_P^0 \leftarrow \phi_P$ and proceed to the next time level
  - To monitor whether steady state reached:

  if $\phi_P - \phi_P^0 < \varepsilon \rightarrow$ steady state . (ε may be 10^{-9})

The Analytical Solution

- Under the given boundary conditions the solution to the problem is as follows:

$$\phi(x) = C_1 + C_2 e^{Px} - \frac{a_0}{P^2}\left(Px+1\right) - \sum_{n=1}^{\infty} a_n \left(\frac{L}{n\pi}\right)\left[P\sin\left(\frac{n\pi x}{L}\right) + \left(\frac{n\pi}{L}\right)\cos\left(\frac{n\pi x}{L}\right) \Big/ \left[P^2 + \left(\frac{n\pi}{L}\right)^2\right]\right] \qquad (8.41)$$

with
$$P = \frac{\rho u}{\Gamma}; \quad C_2 = \frac{a_0}{P^2 e^{PL}} + \sum_{n=1}^{\infty} \frac{a_n}{e^{PL}} \cos\left(n\pi\right) \Big/ \left[P^2 + \left(\frac{n\pi}{L}\right)^2\right]$$

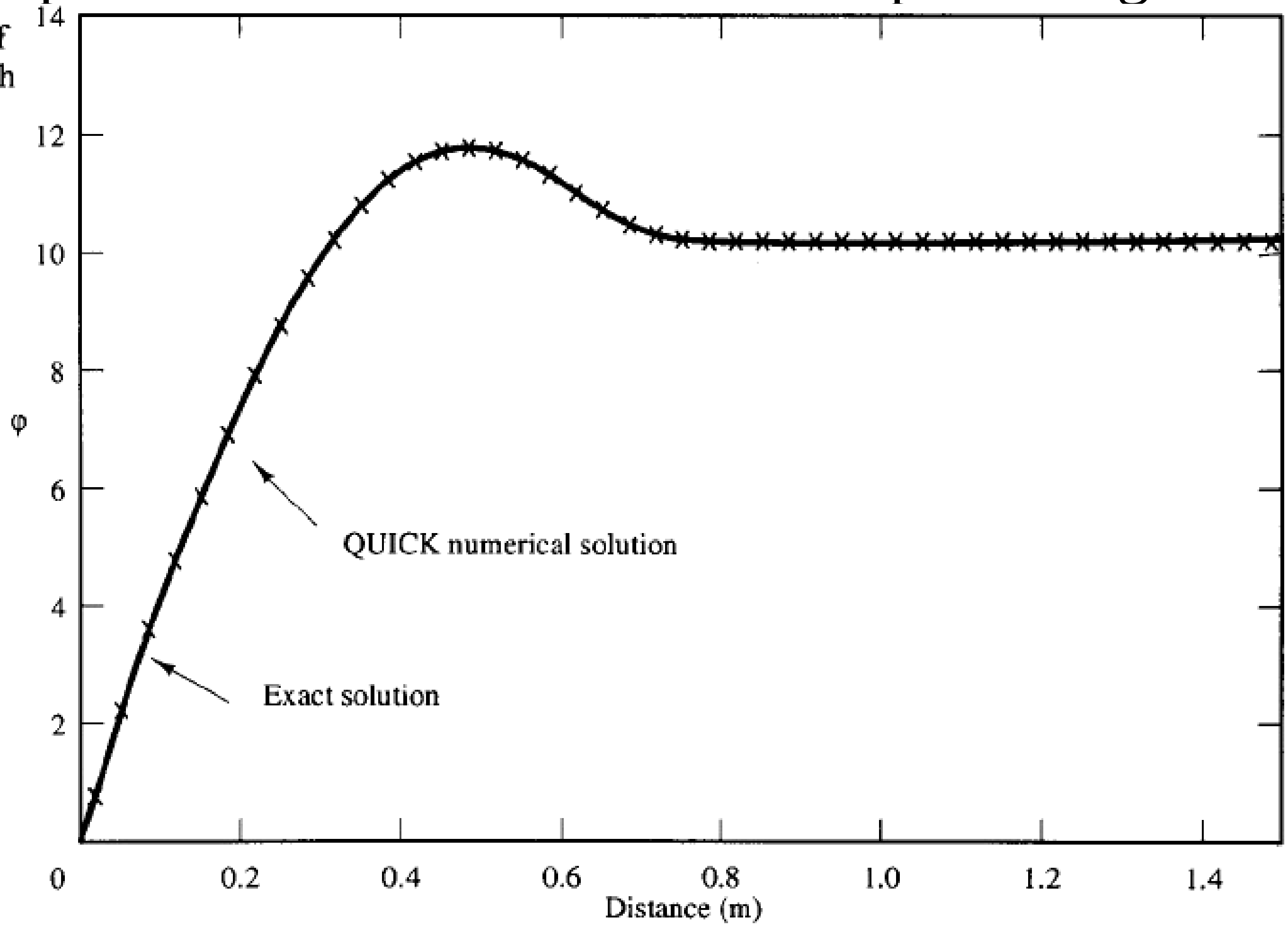
and
$$C_1 = -C_2 + \frac{a_0}{P^2} + \sum_{n=1}^{\infty} a_n \Big/ \left[P^2 + \left(\frac{n\pi}{L}\right)^2\right]$$

and
$$a_0 = \frac{\left(x_1 + x_2\right)\left(ax_1 + b\right) + bx_1}{2L}$$

$$a_n = \frac{2L}{n^2\pi^2}\left\{\left(\frac{a\left(x_1+x_2\right)+b}{x_2}\right)\cos\left(\frac{n\pi x_1}{L}\right) - \left[a + \left(\frac{ax_1+b}{x_2}\right)\cos\left(\frac{n\pi\left(x_1+x_2\right)}{L}\right)\right]\right\}$$

- The analytical and numerical steady state solutions are compared in Figure 8.9. as can be seen the use of the QUICK scheme and a fine grid for spatial discretisation ensure near-perfect agreement

Fig. 8.9 Comparison of the numerical results with the analytical solution



8.7.1 Transient SIMPLE

The continuity equation in a transient two-dimensional flow is given by

$$\frac{\partial \rho}{\partial t}+\frac{\partial(\rho u)}{\partial x}+\frac{\partial(\rho v)}{\partial y}=0 \qquad (8.42)$$

The integrated form of this eqn over a two-dimensional scalar CV is

$$\frac{\left(\rho_P-\rho_P^0\right)}{\Delta t}\Delta V+\left[(\rho u A)_e-(\rho u A)_w\right]+\left[(\rho u A)_n-(\rho u A)_s\right]=0 \qquad (8.43)$$

The equivalent of pressure correction equation (5.32) for a two-dimensional transient flow will take the form

$$a_P p'_P = a_W p'_W + a_E p'_E + a_S p'_S + a_N p'_N + b$$

$$a_E=(\rho A d)_e \quad a_W=(\rho A d)_w \quad a_N=(\rho A d)_n \quad a_S=(\rho A d)_s \qquad (8.44)$$

$$a_P = a_w + a_e + a_s + a_n$$

$$b=\left(\rho u^* A\right)_w-\left(\rho u^* A\right)_e+\left(\rho v^* A\right)_s-\left(\rho v^* A\right)_n+\frac{(\rho_P^o-\rho_P)\Delta V}{\Delta t}$$

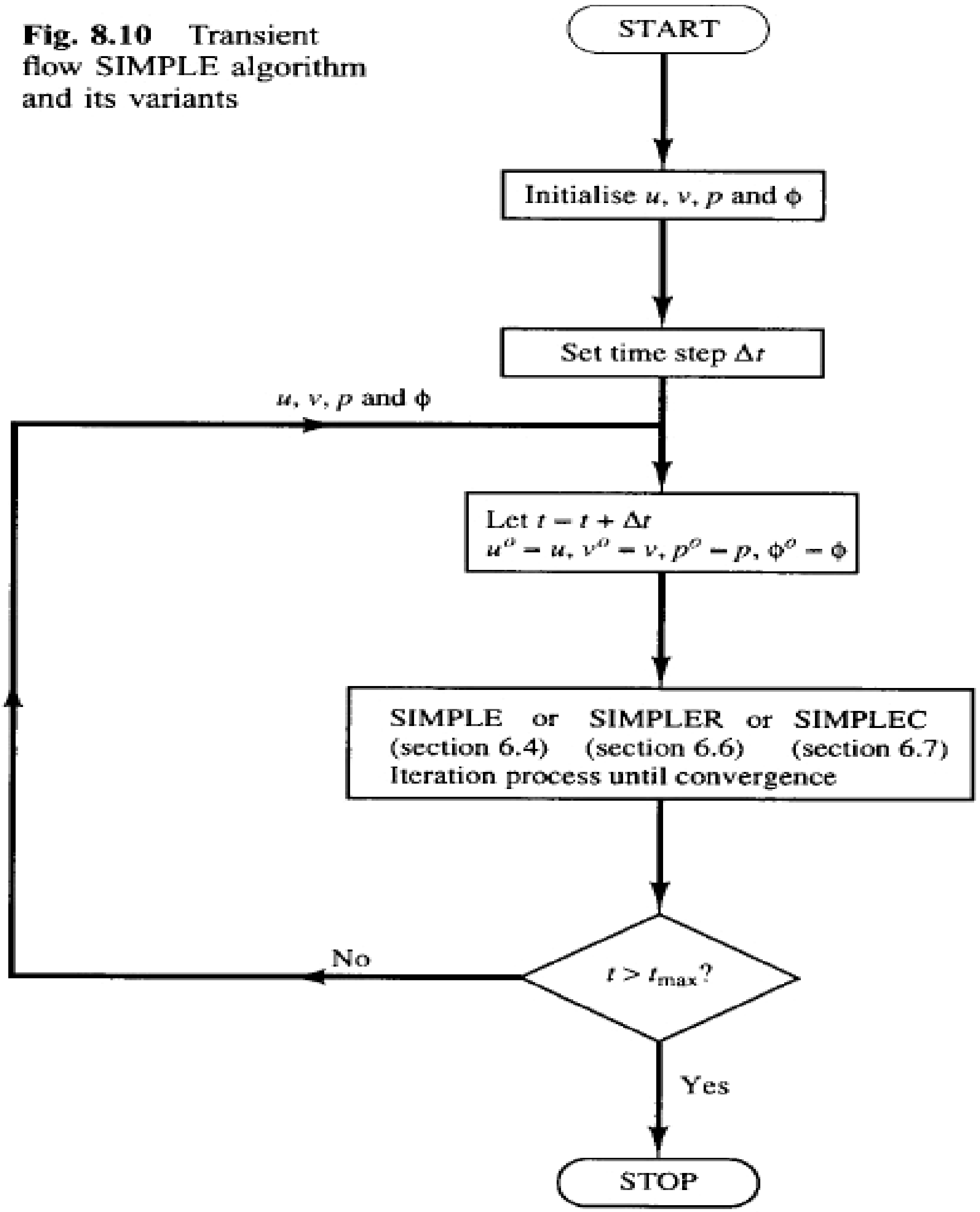
A higher order differencing scheme may also be used for time derivative

A second order accurate scheme with respect to time is

$$\frac{\partial T}{\partial t} = \frac{1}{2\Delta t}\left(3T^{n+1} - 4T^{n} + T^{n-1}\right) \qquad (8.45)$$

T^n and T^{n-1} are known from previous time steps → they are treated as source terms and are placed on the rhs of the equation.

Fig. 8.10 Transient flow SIMPLE algorithm and its variants



8.8 Steady state calculations using the pseudo-transient approach

- It was mentioned in chapter 6 that under-relaxation is necessary to stabilize the iterative process of obtaining steady state solutions. The under-relaxed form of the two-dimensional u-momentum equation, for example, takes the form

$$\frac{a_{i,J}}{\alpha_u} u_{i,J} = \sum a_{nb} u_{nb} + \left(p_{I-1,J} - p_{I,J} \right) A_{i,J} + b_{i,J} + \left[\left(1 - \alpha_u\right) \frac{a_{i,J}}{\alpha_u} \right] u_{i,J}^{(n-1)} \qquad (8.46)$$

- Compare this with the transient (implicit) u-momentum equation

$$\left(a_{i,J} + \frac{\rho_{i,J}^{0} \Delta V}{\Delta t} \right) u_{i,J} = \sum a_{nb} u_{nb} + \left(p_{I-1,J} - p_{I,J} \right) A_{i,J} + b_{i,J} + \frac{\rho_{i,J}^{0} \Delta V}{\Delta t} u_{i,J}^{0} \qquad (8.47)$$

- In equation (8.46) the superscript n-1 indicates the previous iteration and in equation(8.47) superscript 0 represents the previous time level. We immediately note a clear analogy between transient calculations and under-relaxation in steady state calculations. It can be easily deduced that

$$\left(1-\alpha_u\right)\frac{a_{i,J}}{\alpha_u}=\frac{\rho^0_{i,J}\Delta V}{\Delta t} \tag{8.48}$$

- This formula shows that it is possible to achieve the effects of under-relaxed iterative steady state calculations from a given initial field by means of a pseudo-transient computation starting from the same initial field by taking a step size that satisfies (8.48).