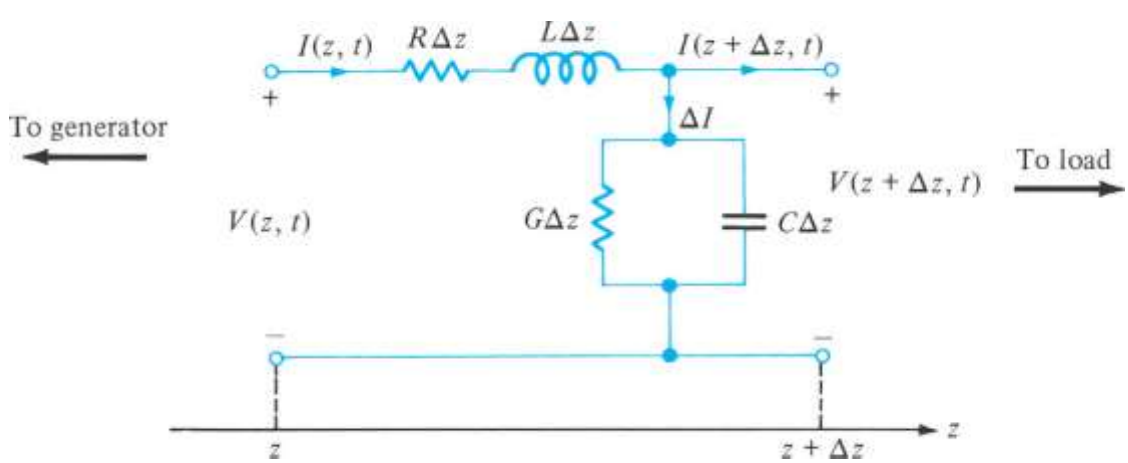
TRANSMISSION LINE THEORY

(TEM Line)

A uniform transmission line is defined as the one whose dimensions and electrical properties are identical at all planes transverse to the direction of propagation.

Circuit Representation of TL's

A uniform TL may be modeled by the following circuit representation:



R**:** Series resistance per unit length of line (for both conductors (ohm/m)).

L**:** Series inductance per unit length of line (Henry/m).

G**:** Shunt conductance per unit length of line (mho/m).

C: Shunt capacitance per unit length of line (Farad/m).

The line is pictured as a cascade of identical sections, each of Δz long.

Since Δz can always be chosen small compared to the operating wavelength, an individual section of line may be analyzed using ordinary ac circuit theory. In the following analysis, we let $\Delta z \to 0$, so the results are valid at all frequencies (hence for any physical time variation).

Applying the Kirchhoff's voltage law to the line section gives:

$$v(z,t) = (R\Delta z)\, i(z,t) + (L\Delta z)\frac{\partial i(z,t)}{\partial t} + v(z+\Delta z,t)$$

Rearranging yields:

$$\frac{v(z+\Delta z,t) - v(z,t)}{\Delta z} = -R\, i(z,t) - L\frac{\partial i(z,t)}{\partial t}$$

Letting $\Delta z \rightarrow 0$, we get,

$$\frac{\partial v(z,t)}{\partial z} = -R\, i(z,t) - L\frac{\partial i(z,t)}{\partial t}$$

Now applying Kirchhoff's current law to the line section gives:

$$i(z,t) = \left(G\Delta z\right) v\left(z+\Delta z,t\right) + \left(C\Delta z\right)\frac{\partial v(z+\Delta z,t)}{\partial t} + i(z+\Delta z,t)$$

Rearranging yields:

$$\frac{i(z+\Delta z,t) - i(z,t)}{\Delta z} = -G\, v(z+\Delta z,t) - C\frac{\partial v(z+\Delta z,t)}{\partial t}$$

Letting $\Delta z \rightarrow 0$

$$\frac{\partial i(z,t)}{\partial z} = -G\, v(z,t) - C\frac{\partial v(z,t)}{\partial t}$$

Then the time domain TL or telegrapher equations are:

$$\frac{\partial v(z,t)}{\partial z} = -R\, i(z,t) - L\frac{\partial i(z,t)}{\partial t}$$

$$\frac{\partial i(z,t)}{\partial z} = -G\, v(z,t) - C\frac{\partial v(z,t)}{\partial t}$$

The solution of these equations, together with the electrical properties of the generator and load, allow us to determine the instantaneous voltage and current at any time *t* and any place *z* along the uniform TL.

Lossless Line: For the case of perfect conductors (R=0) and insulators (G=0), the telegrapher equations reduce to the following form:

$$\frac{\partial v(z,t)}{\partial z} = -L\frac{\partial i(z,t)}{\partial t}$$

$$\frac{\partial i(z,t)}{\partial z} = -C\frac{\partial v(z,t)}{\partial t}$$

$$\frac{\partial}{\partial z}\left(\frac{\partial v(z,t)}{\partial z}\right) = -L\frac{\partial}{\partial z}\left(\frac{\partial i(z,t)}{\partial t}\right)$$

$$-L\frac{\partial}{\partial t}\left(\frac{\partial i(z,t)}{\partial z}\right) = -L\frac{\partial}{\partial t}\left(-C\frac{\partial v}{\partial t}\right)$$

Or,

$$\frac{\partial^2 v(z,t)}{\partial z^2} - LC\frac{\partial^2 i(z,t)}{\partial t^2} = 0$$

$$\frac{\partial^2 i(z,t)}{\partial z} - LC\frac{\partial^2 i(z,t)}{\partial t^2} = 0$$

Wave equation's for voltage and current on a lossless TL.

Although real lines are never lossless, lolessness approximation for practical TL's is very usefull.

TRANSMISSION LINES WITH SINUSOIDAL EXCITATION

We will only consider the sinusoidal steady-state solutions.

Transmission-Line Equations:

Under sinusoidal steady state conditions, the TL equations take the form:

$$\frac{dV(z)}{dz} = -(R + j\omega L)I(z)$$

$$\frac{dI(z)}{dz} = -(G + j\omega C)V(z)$$

Where $V(z)$ and $I(z)$ are voltage and current phasors.

The real sinusoidal voltage and current waveforms are obtained from:

$$v(z,t) = \text{Re}\left[V(z)e^{j\omega t}\right]$$

$$i(z,t) = \text{Re}\left[I(z)e^{j\omega t}\right]$$

Wave Propagation on a TL

The second order differential equations for $V(z)$ and $I(z)$ are:

$$\frac{d^2V(z)}{dz^2} - \gamma^2 V(z) = 0$$

$$\frac{d^2I(z)}{dz^2} - \gamma^2 I(z) = 0$$

where $\gamma = \alpha + j\beta = \left[(R + j\omega L)(G + j\omega C) \right]^{1/2}$

$\gamma =$ complex propagation constant.

$\alpha =$ attenuation constant (Np/m).

$\beta =$ phase constant (rad/m).

The solution for $V(z)$ is:

$$V(z) = V_0^+ e^{-\gamma z} + V_0^- e^{+\gamma z}$$

$$V(z) = V^+(z) + V^-(z)$$

Where V_0^+ and V_0^- are constants independent of *z*, and

$$V(z) = V_0^+ e^{-\gamma z}$$

$$V(z) = V_0^- e^{+\gamma z}$$

Represent voltage waves traveling on the line in the positive and negative z directions respectively.

WAVE PROPAGATION ON A TL

The second order equations for *V(z)* and *I(z)* are:

$$\frac{d^2V(z)}{dz^2} - \gamma^2 V(z) = 0$$

$$\frac{d^2I(z)}{dz^2} - \gamma^2 I(z) = 0$$

The solution for *V(z)* is:

$$V(z) = V_0^+ e^{-\gamma z} + V_0^- e^{+\gamma z}$$

Where V_0^+ is a constant of voltage waves traveling in the forward direction, V_0^- is a constant of voltage waves traveling in the reverse direction, independent of z.

Now consider the equation:

$$\frac{dV(z)}{dz} = -(R + j\omega L) I(z)$$

If we substitude the solution for *V(z)* into the above equation we get,

$$-\gamma V_0^+ e^{-\gamma z} + \gamma V_0^- e^{\gamma z} = -(R + j\omega L) I(z)$$ or,

$$I(z) = \frac{\gamma}{(R + j\omega L)} V_0^+ e^{-\gamma z} - \frac{\gamma}{(R + j\omega L)} V_0^- e^{\gamma z}$$

$$\frac{\gamma}{(R + j\omega L)} = \frac{\left[\left(R + j\omega L\right)(G + j\omega C)\right]^{1/2}}{\left(R + j\omega L\right)}$$

$$\frac{\gamma}{(R + j\omega L)} = \left(\frac{(G + j\omega C)}{\left(R + j\omega L\right)}\right)^{1/2}$$

Define

$$Z_0 = \left(\frac{(R + j\omega L)}{(G + j\omega C)} \right)^{1/2}$$

the characteristic impedance of the TL. Then,

$$\frac{\gamma}{(R + j\omega L)} = \frac{1}{Z_0}$$

So,

$$I(z) = \frac{V_0^+}{Z_0} e^{-\gamma z} - \frac{V_0^-}{Z_0} e^{\gamma z} = I_0^+ e^{-\gamma z} - I_0^- e^{+\gamma z}$$

$$= I^+(z) - I^-(z)$$

Where,

$$I_0^+ = \frac{V_0^+}{Z_0} \qquad\qquad I_0^- = \frac{V_0^-}{Z_0}$$

Constants independent of z.

$$I^+(z) = \frac{V_0^+}{Z_0} e^{-\gamma z} \quad I^-(z) = \frac{V_0^-}{Z_0} e^{+\gamma z}$$

Lossless Line:

For the lossless line R=0, G=0.

i) $$Z_0 = \sqrt{\frac{L}{C}} \quad \Omega$$

ii)
$$\begin{aligned} \gamma &= \sqrt{(R + j\omega L)(G + j\omega C)} \\ &= \sqrt{(j\omega L)(j\omega C)} = j\omega LC \\ &= j\beta \end{aligned}$$

$\beta = \omega\sqrt{LC}$ Propagation constant

$\alpha = 0$ (no loss)

iii)

$$V(z) = V_0^+ e^{-j\beta z} + V_0^- e^{+j\beta z}$$

$$I(z) = \frac{1}{Z_0}(V_0^+ e^{-j\beta z} - V_0^- e^{+j\beta z})$$

iv) If $V_0^+ = \left|V_0^+\right| e^{j\theta_+}$ $\quad V_0^- = \left|V_0^-\right| e^{j\theta_-}$ then

$$v(z,t) = \left|V_0^+\right| \cos\left(\omega t - \beta z + \theta_+\right) + \left|V_0^-\right| \cos\left(\omega t + \beta z + \theta_-\right)$$

$$i(z,t) = \frac{1}{Z_0}\left[\left|V_0^+\right| \cos\left(\omega t - \beta z + \theta_+\right) - \left|V_0^-\right| \cos\left(\omega t + \beta z + \theta_-\right)\right]$$

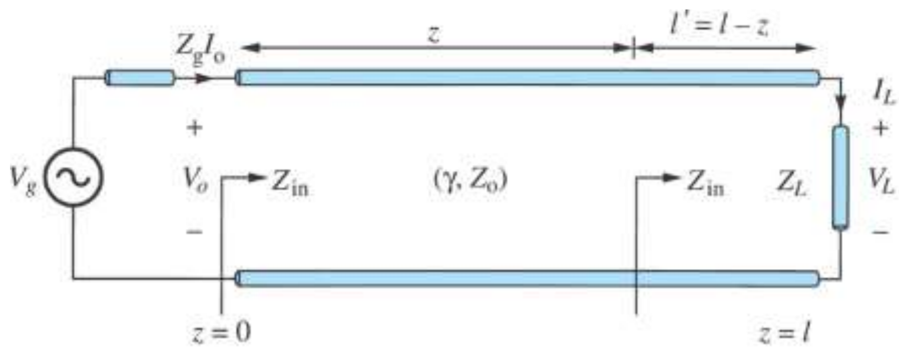
v) $\lambda = \dfrac{2\pi}{\beta} = \dfrac{2\pi}{2\pi f\sqrt{LC}} = \dfrac{1}{f\sqrt{LC}}$

vi) $u_p = \dfrac{\omega}{\beta} = \dfrac{\omega}{\omega\sqrt{LC}} = \dfrac{1}{\sqrt{LC}}$ So,

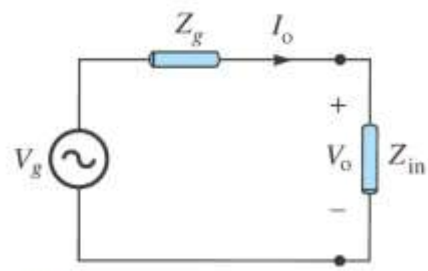
$$u_p = \frac{1}{\sqrt{LC}} \qquad \lambda = \frac{v_p}{f}$$

$$u_p = \lambda f$$

TERMINATED TRANSMISSION LINE



(a)



(b)

Infinitely Long TL

For the infinite line, only forward traveling waves exist and therefore at z=0,

$$V = V_0^+ \qquad I = \frac{V_0^+}{Z_0} = I_0^+$$

$$Z_{in} = \frac{V}{I} = Z_0$$

Suppose now that the infinite line is broken at $z = \ell$. Since the line to the right of $z = \ell$ is still infinite, its input impedance is Z_0 and therefore replacing it by a load impedance of the same value does not change any of the conditions to the left of $z = \ell$. This means that a <u>finite line terminated in its characteristic impedance</u> is equivalent to an infinitely long line. Like the infinite line, a finite length line, terminated in Z_0 has no reflections. Also, its input impedance is equal to Z_0 and independent of the line length.

$$V_{in} = V_0 = \frac{Z_0}{Z_0 + Z_g} V_g$$

$$I_{in} = I_0 = \frac{V_g}{Z_0 + Z_g}$$

$$V_{in} = V_0^+$$

$$I_{in} = I_0^+ = \frac{V_0^+}{Z_0}$$

So,

$$V_0^+ = \frac{Z_0}{Z_0 + Z_g} V_g$$

And

$$V(z) = \frac{Z_0}{Z_0 + Z_g} V_g \, e^{-\alpha z} e^{-j\beta z}$$

$$I(z) = \frac{V_g}{Z_0 + Z_g} e^{-\alpha z} e^{-j\beta z}$$

The time-averaged power absorbed by the load is:

$$P_L = \frac{1}{2}\mathrm{Re}\left(V_L I^*{}_L\right) = \frac{1}{2}\mathrm{Re}\left[\frac{Z_0}{Z_0 + Z_g} V_g e^{-\alpha l} e^{-j\beta l}\left(\frac{V_g}{Z_0 + Z_g}\right)^* e^{-\alpha l} e^{j\beta l}\right]$$

$$P_L = \frac{Z_0}{2} e^{-2\alpha l} \frac{\left|V_g\right|^2}{\left|Z_0 + Z_g\right|^2} = \frac{1}{2} Z_0 e^{-2\alpha l}\left|\frac{V_g}{Z_0 + Z_g}\right|^2$$

If $\alpha = 0$ (lossless line) $\quad P_L = \frac{1}{2} Z_0 \left|\frac{V_g}{Z_0 + Z_g}\right|^2 = P_{in}$